Máster Universitario en Administración y Dirección de Empresas

Full Time MBA

Quantitative methods for decision making

Professor Andrea Saltelli

upf. BARCELONA SCHOOL OF MANAGEMENT

Elements of quantification for decision making with emphasis on operation research

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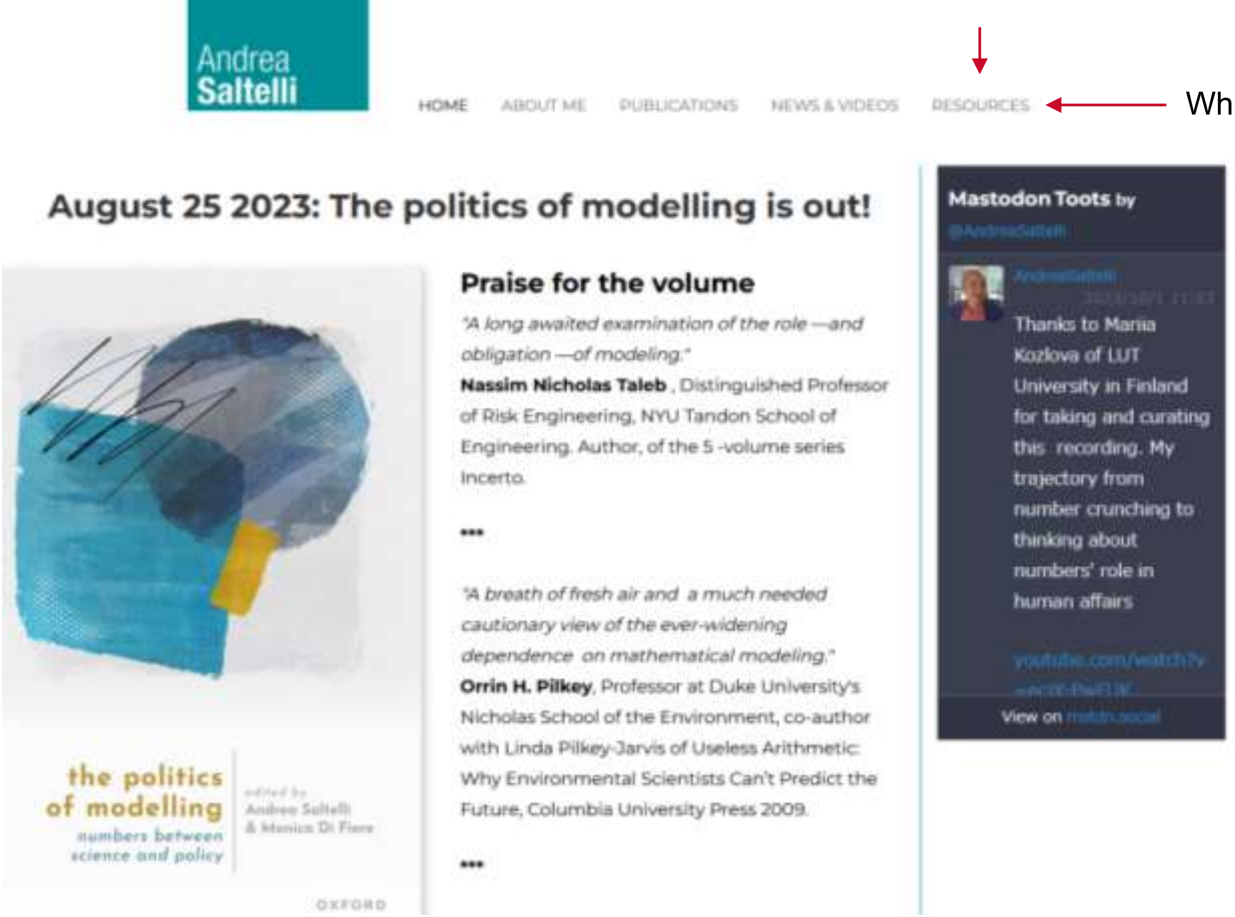

August 25 2023: The politics of modelling is out!

Praise for the volume

"A long awaited examination of the role —and obligation —of modeling."
Nassim Nicholas Taleb , Distinguished Professor of Risk Engineering, NYU Tandon School of Engineering. Author, of the 5 -volume series Incerto.

•••

"A breath of fresh air and a much needed cautionary view of the ever-widening dependence on mathematical modeling."
Orrin H. Pilkey, Professor at Duke University's Nicholas School of the Environment, co-author with Linda Pilkey-Jarvis of Useless Arithmetic: Why Environmental Scientists Can't Predict the Future, Columbia University Press 2009.

•••

Mastodon Toots by
@AndreaSaltelli

AndreaSaltelli

Thanks to Mariia Kozlova of LUT University in Finland for taking and curating this recording. My trajectory from number crunching to thinking about numbers' role in human affairs

youtube.com/watch?v

View on mstdn.social

Where to find this talk

The talk is also at

https://ecampus.bsm.upf.edu/,

where you find additional reading material

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Notes from previous homework

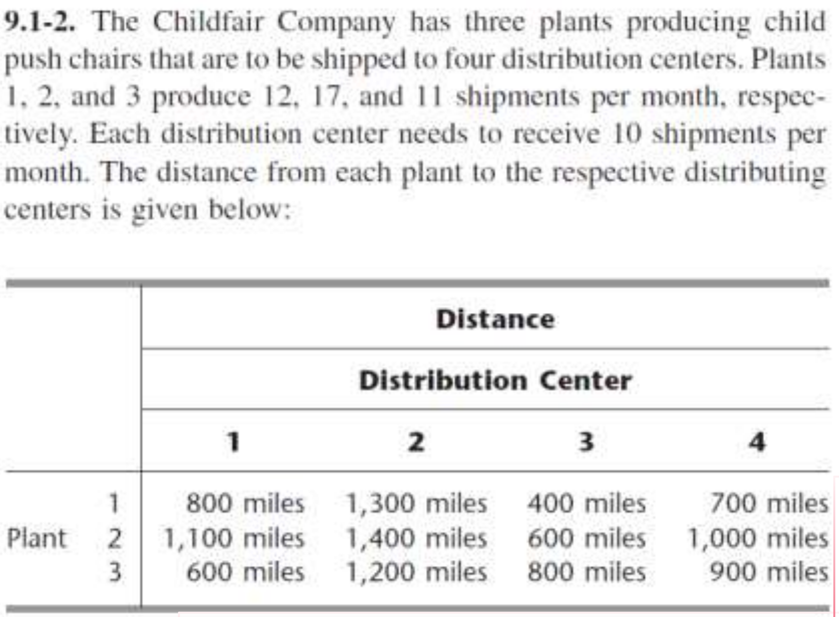
Please use ‘official’ forms, four faces, don’t split them; don’t scribble your computations on the form, please use separate scratch paper

Rounding: 165/345?

$165/345 = .476878\cdots$ rounds up to .477 not 476

Homework

1) Write the equations for this transportation problem knowing that freight cost for each shipment is $100 plus 50 cents per mile. How much should be shipped from each plant to each of the distribution centers to minimize the total shipping cost?



9.1-2. The Childfair Company has three plants producing child push chairs that are to be shipped to four distribution centers. Plants 1, 2, and 3 produce 12, 17, and 11 shipments per month, respectively. Each distribution center needs to receive 10 shipments per month. The distance from each plant to the respective distributing centers is given below:

		Distance				
		Distribution Center				
		1	**2**	**3**	**4**	
Plant	1	800 miles	1,300 miles	400 miles	700 miles	12
	2	1,100 miles	1,400 miles	600 miles	1,000 miles	17
	3	600 miles	1,200 miles	800 miles	900 miles	11
		10	10	10	10	

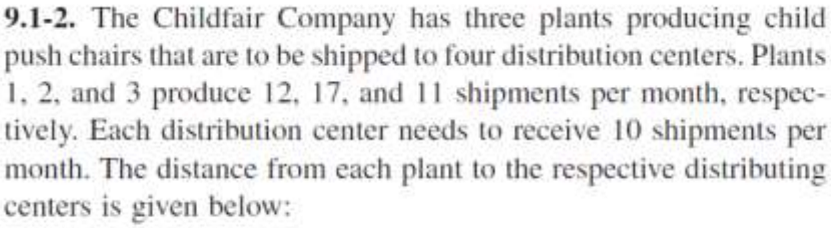
Homework
1) Write the equations for this transportation problem knowing that freight cost for each shipment is $100 plus 50 cents per mile. How much should be shipped from each plant to each of the distribution centers to minimize the total shipping cost?

$$x_1 + x_2 + x_3 + x_4 = 12$$
$$x_5 + x_6 + x_7 + x_8 = 17$$
$$x_9 + x_{10} + x_{11} + x_{12} = 11$$

Plant constraints

$$x_1 + x_5 + x_9 = 10$$
$$x_2 + x_6 + x_{10} = 10$$
$$x_3 + x_7 + x_{11} = 10$$
$$x_4 + x_8 + x_{12} = 10$$

Distribution centres constraints



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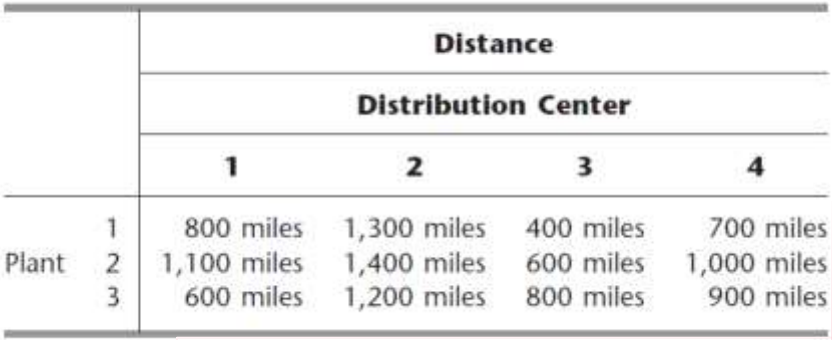

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	3	600 miles	1,200 miles	800 miles	900 miles

12
17
11

10 10 10 10

$$\text{Minimize } Z = (100.+.5 * 800)x_1 + (100.+.5 * 1{,}300)x_2 + (100.+.5 * 400)x_3 + (100.+.5 * 700)x_4 + [\text{same for rows two and three all the way to }] \ldots + (100.+.5 * 900)x_{12}$$

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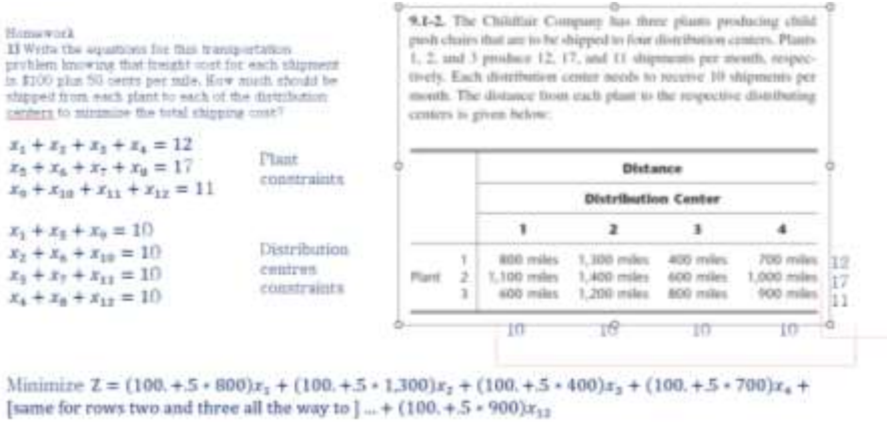

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Minimize $Z = (100. + .5 \cdot 800)x_1 + (100. + .5 \cdot 1{,}300)x_2 + (100. + .5 \cdot 400)x_3 + (100. + .5 \cdot 700)x_4 +$ [same for rows two and three all the way to] ... $+ (100. + .5 \cdot 900)x_{12}$

Solution with Excel Solver

Z=16,300

See it at

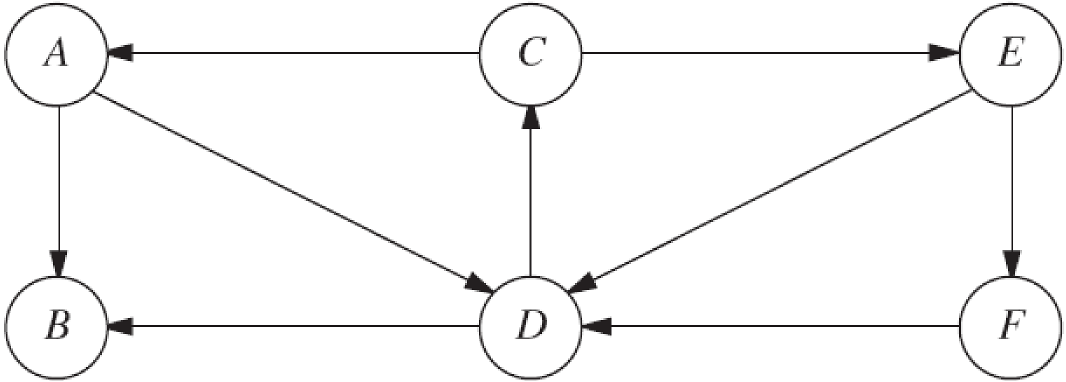
https://www.andreasaltelli.eu/file/repository/Childfair_Company.xlsx

and Ecampus

Decision var. & Constraints		Dist. Centers						
		1	2	3	4	Shipments		Available
Plant	1	0	**2**	**10**	0	12	<=	12
	2	0	**7**	0	**10**	17	<=	17
	3	**10**	**1**	0	0	11	<=	11
Received		10	10	10	10			
		>=	>=	>=	>=			
Required		10	10	10	10			

Homework

2) Consider the following directed network (Hillier 9.2-1)

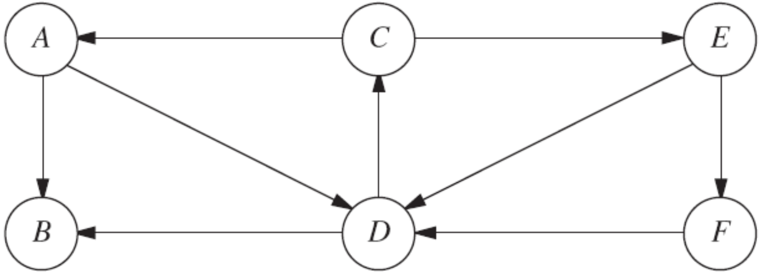


(a) Find a directed path from node A to node F, and then identify three other undirected paths from node A to node F.

(b) Find three directed cycles. Then identify an undirected cycle that includes every node.

(c) Identify a set of arcs that forms a spanning tree.

(d) Use the process illustrated in Fig. 10.3 to grow a tree one arc at a time until a spanning tree has been formed. Then repeat this process to obtain another spanning tree. [Do not duplicate the spanning tree identified in part (c).]



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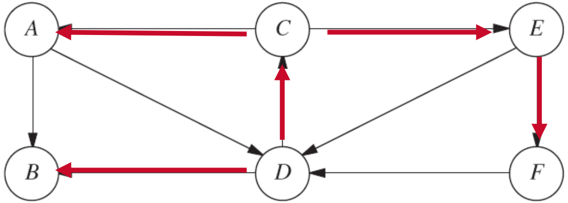
(a) $A \to D \to C \to E \to F$ directed

$A \to C \to D \to F$ undirected
$A \to B \to D \to F$ undirected
$A \to D \to F$ undirected

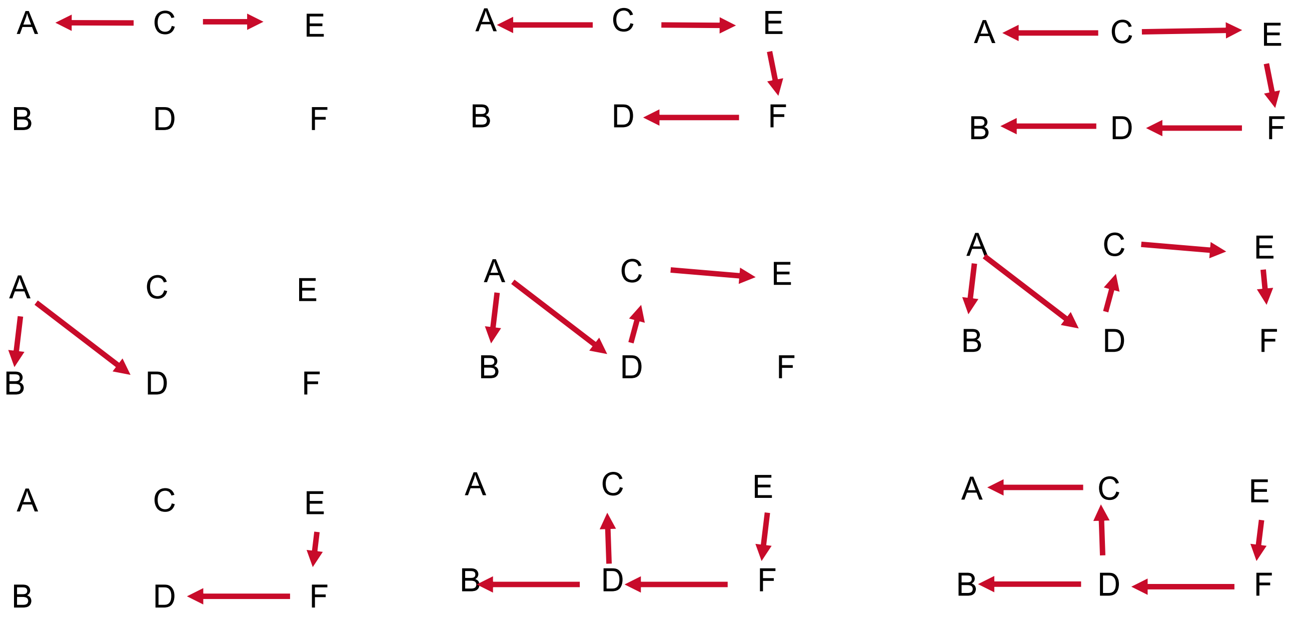
(b) $A \to D \to C \to A$ directed
$D \to C \to E \to D$ directed
$E \to F \to D \to C \to E$ directed

$A \to B \to D \to F \to E \to C \to A$ undirected all nodes

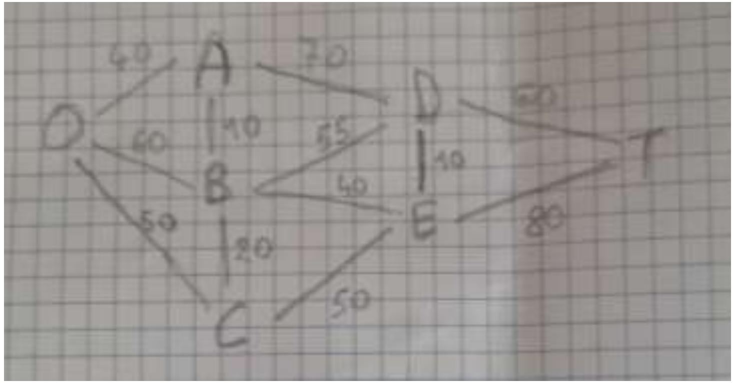
(c) Spanning tree



(c) Three different spanning trees



Homework 3) You need to take a trip by car to another town that you have never visited before. Therefore, you are studying a map to determine the shortest route to your destination. Depending on which route you choose, there are five other towns (call them A, B, C, D, E) that you might pass through on the way. The map shows the mileage along each road that directly connects two towns without any intervening towns. These numbers are summarized in the following table, where a dash indicates that there is no road directly connecting these two towns without going through any other towns. Formulate this problem as a shortest-path problem by drawing a network where nodes represent towns, links represent roads, and numbers indicate the length of each link in miles.



	Miles between Adjacent Towns					
Town	**A**	**B**	**C**	**D**	**E**	**Destination**
Origin	40	60	50	—	—	—
A		10	—	70	—	—
B			20	55	40	—
C				—	50	—
D					10	60
E						80

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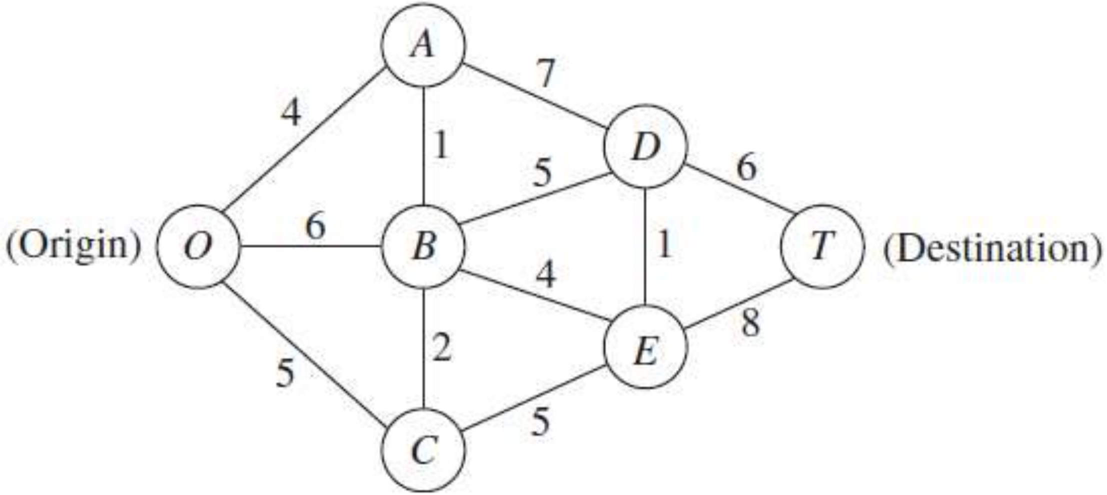
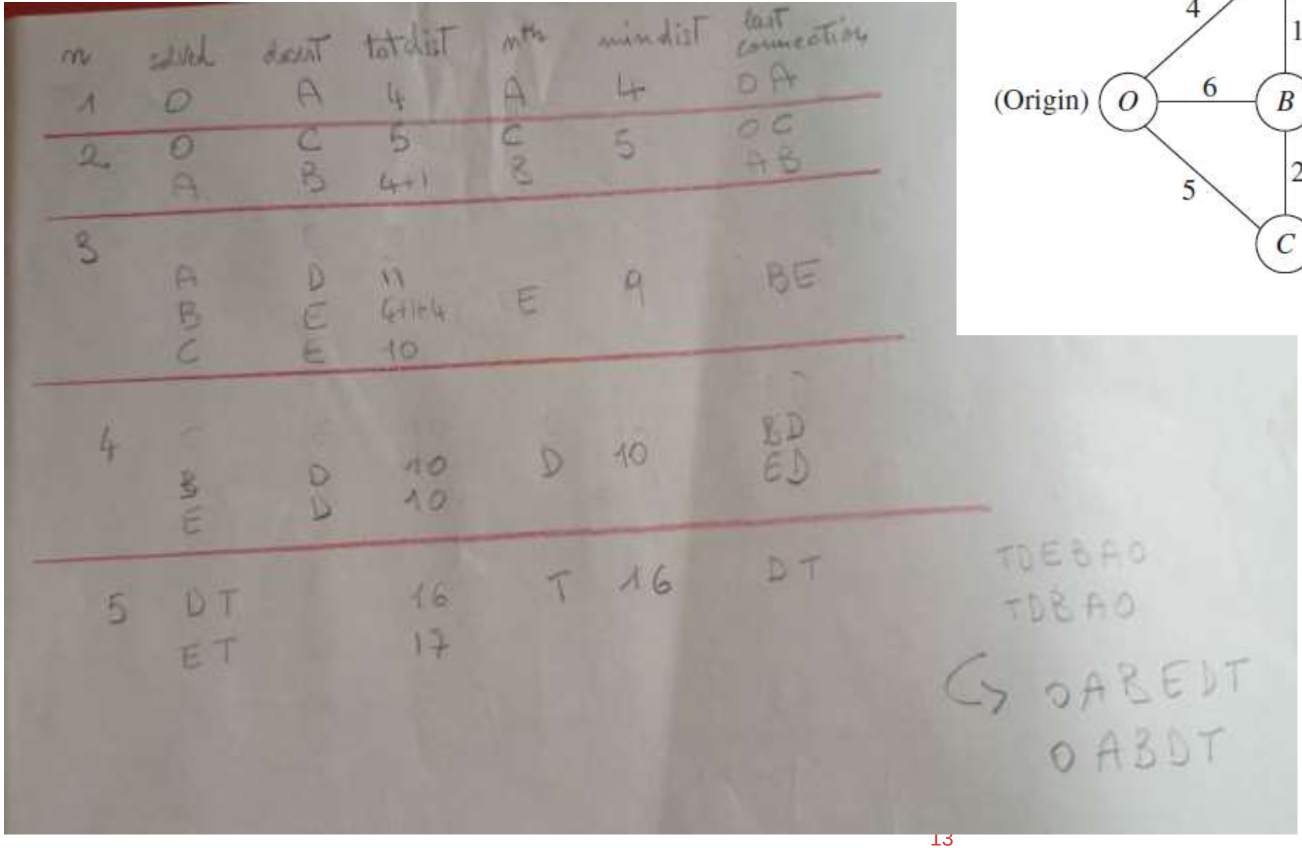
Homework **4)** Find shortest path from ***O*** to ***T*,** first visually then using the table method and backward recursion studied in Lesson 4 (Hillier 10.3-4); the first row of the table in give below.



n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	4	*A*	4	*OA*

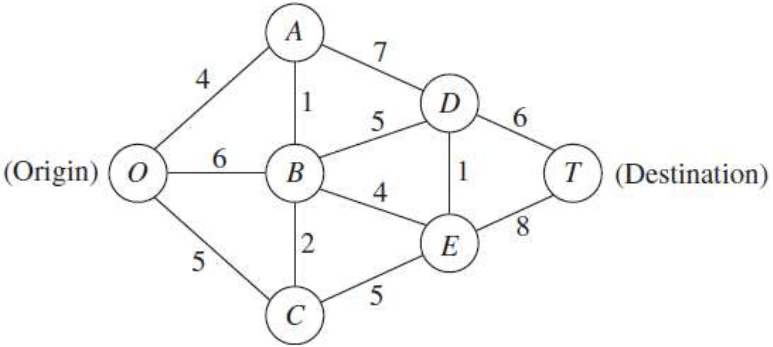
Homework 4)



n	solved	dest	totdist	nth	mindist	last connection
1	O	A	4	A	4	OA
2	O	C	5	C	5	OC
	A	B	4+1	B		AB
3	A	D	11		9	BE
	B	E	6+1+4	E		
	C	E	10			
4						BD
	B	D	10	D	10	ED
	E	D	10			
5	DT		16	T	16	DT
	ET		17			

TDEBAO
TDBAO

↳ OABEDT
OABDT



In this set of slides:

15 Integer programming (continued)

16 Nonlinear programming

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15.

Integer Programming (continued)

Solution via branch and bound. Take home points.
Hillier 2014, chapter 12.

More tricks with binary variables. From Hillier 10th edition, section 12.3 INNOVATIVE USES OF BINARY VARIABLES IN MODEL FORMULATION

When one of two constraints must hold, for example

$$3x_1 + 5x_2 - 7x_3 \le 12$$

or

$$4x_1 + 2x_2 + x_3 \le 15$$

But not both we can use an auxiliary binary variable y and impose

$$3x_1 + 5x_2 - 7x_3 \le 12 + My$$
$$4x_1 + 2x_2 + x_3 \le 15 + M(1-y)$$
$$x_i \ge 0$$
$$y \text{ binary}$$

Where M is the usual large number.
If $y = 0$ the first constraint holds, if $y = 1$ the second

More tricks with binary variables.

Another case where IP comes handy is when a problem needs one out of a set of constraints to hold. Old example:

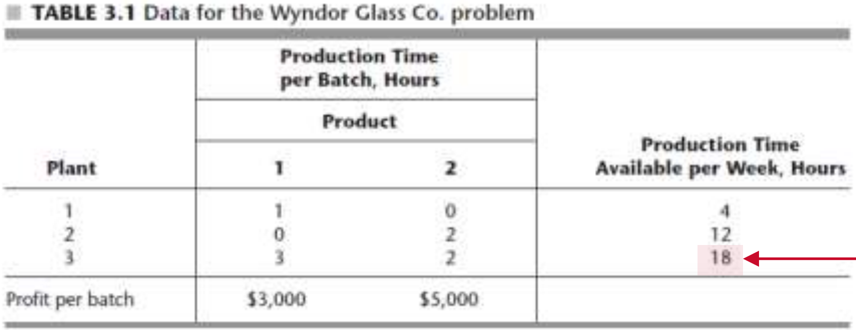

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours Product 1	 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3,000	\$5,000	

OLD PROBLEM

Maximize $Z = 3x_1 + 5x_2$

Subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Imagine we change one constraint so that in plant three there are three possible available times, depending on other possible manufactures so $3x_1 + 2x_2$ must now be equal to either 6 or 12 or 18

Tip: use three binary variable y_1, y_2, y_3

$$3x_1 + 2x_2 = 6y_1 + 12y_2 + 18y_3$$

$y_1 + y_2 + y_3 = 1$ and $\cdots$?

y_1, y_2, y_3 binary

This is now a mixed integer programming problem (MIP)

"It is common for an IP algorithm to begin by applying the simplex method to the LP relaxation to check whether this fortuitous outcome has occurred"

This may or may not work see e.g. the simple example

Maximize $Z = x_2$ (this mean x_2 as high as possible) subject to

$-x_1 + x_2 \leq \frac{1}{2}$

$x_1 + x_2 \leq \frac{7}{2}$

Find graphically the linear solution of this problem

and

$x_1 \geq 0, x_2 \geq 0$

x_1, x_2 integers

I.e. removing this constraint

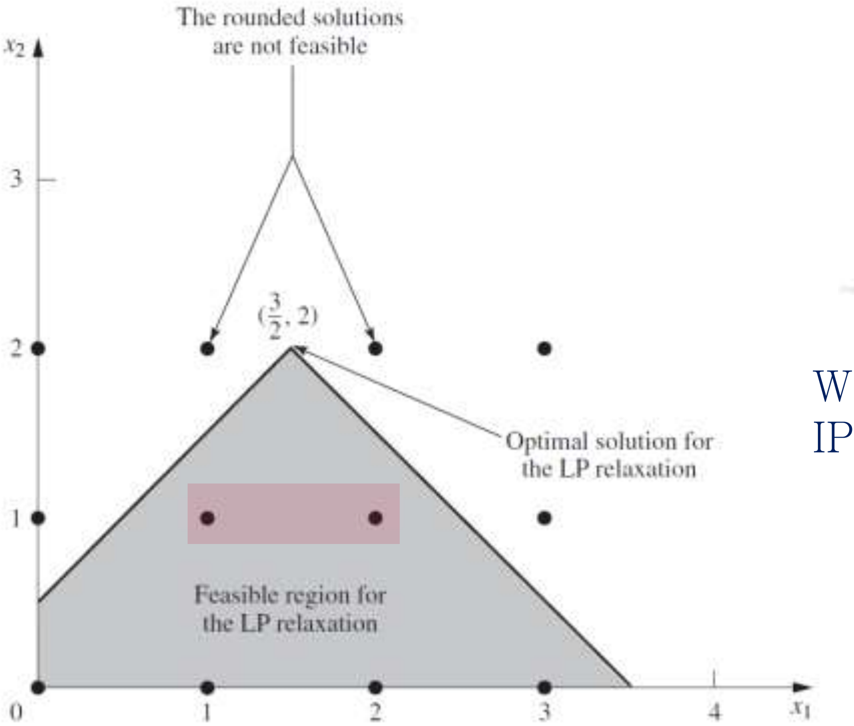

FIGURE 12.2
An example of an IP problem where the optimal solution for the LP relaxation cannot be rounded in any way that retains feasibility.

Which is instead the IP solutions?

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Another case where the relaxation solution can be **not OK**

Maximize $Z = x_1 + 5x_2$ subject to

$x_1 + 10x_2 \leq 20$

$x_1 \leq 2$

and

$x_1 \geq 0, x_2 \geq 0$

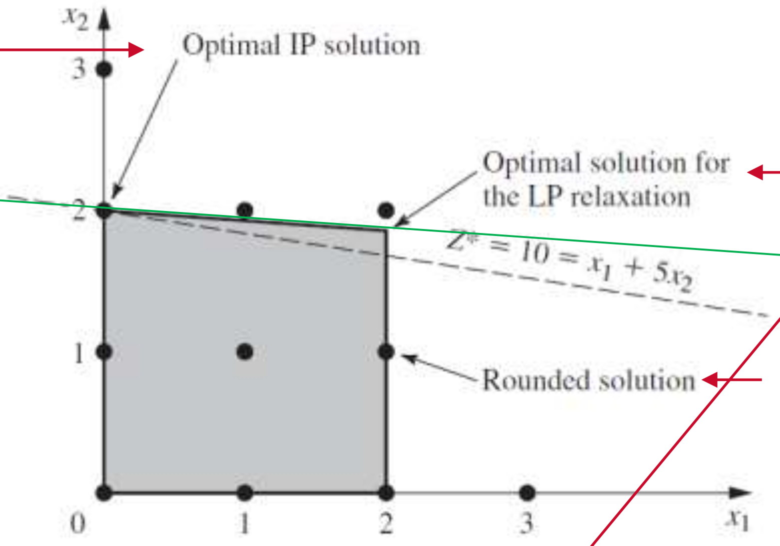
x_1, x_2 integers

Find graphically the <u>linear</u> solution of this problem ⋯

⋯ i.e. removing this constraint

$x_1 = 0,$
$x_2 = 2$
$Z = 10$

Line $x_1 + 10x_2 = 20$



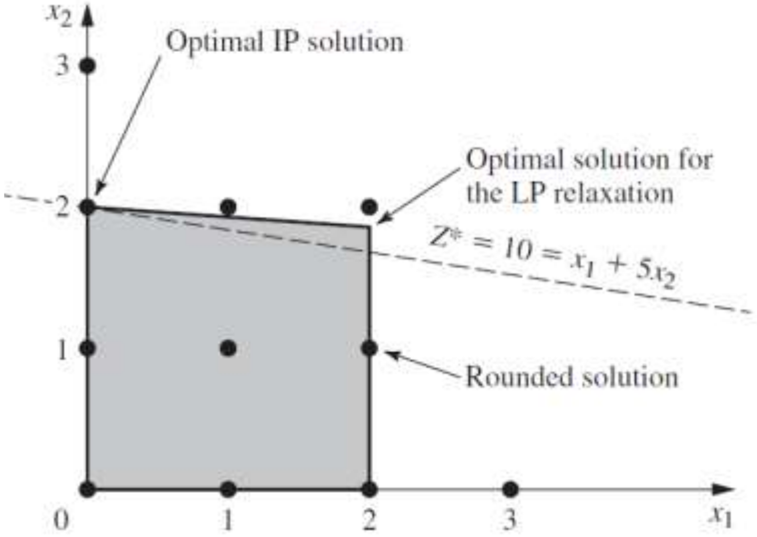
$x_1 = 2,$
$x_2 = \frac{9}{5}$
$Z = 11$

$x_1 = 2,$
$x_2 = 1$
$Z = 7$

FIGURE 12.3
An example where rounding the optimal solution for the LP relaxation is far from optimal for the IP problem.

Did we violate the rule that the LP solution is an upper bound for the IP solution?

When there are may dimensions checking that the relaxation solution is OK can be tricky;



Here we have only **7 integer** points in the feasible region, but the number of points grows exponentially with the number of dimensions →

In many dimensions better use metaheuristic method (such as genetic algorithms, more later) that also work for nonlinear problems.

But there are IP problems whose structure guarantees an integer solution; remember the Transportation Problem (Section 12);

The integer solutions property: For transportation problems where every supply s_i and demand d_i have an integer value, all basic feasible (BF) solutions (including an optimal one) also have integer values

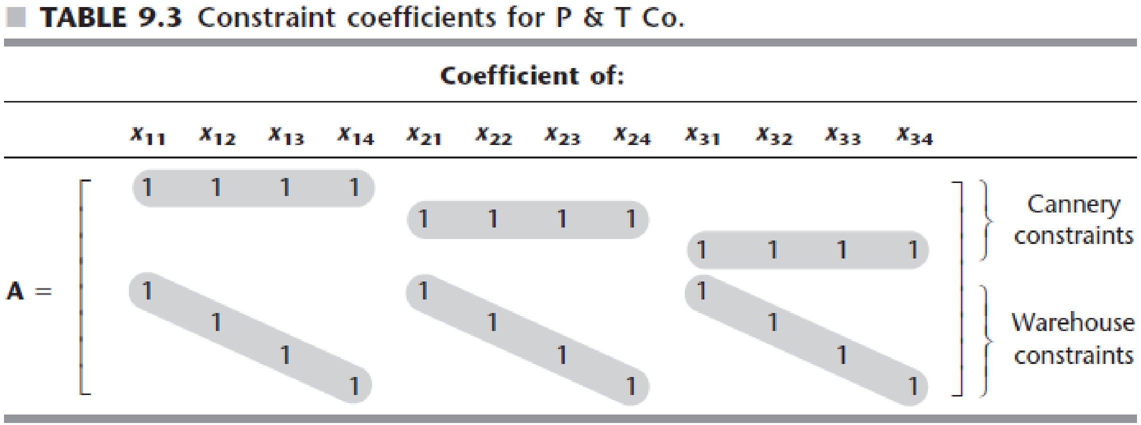


TABLE 9.3 Constraint coefficients for P & T Co.

Coefficient of:

	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	
A =	1	1	1	1									Cannery constraints
					1	1	1	1					
									1	1	1	1	
	1				1				1				Warehouse constraints
		1				1				1			
			1				1				1		
				1				1				1	

But there are IP problems whose structure guarantees an integer solution; remember from the section on Transportation Problem (Section 12);

Source: Wikipedia Commons

Other special cases are the assignment problem, the shortest-path problem, and the maximum flow problem

Charles Chaplin's Modern Times, source http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

Ramon Casas and Pere Romeu on a Tandem, Barcelona. Source: Wikipedia Commons

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Level of difficulty of LP versus IP

	Difficulty of LP problem	Difficulty of IP problem
Source	Number of constraints	Number of integer variables
		Binary or general integer?
		Special form?

Source: https://www.dreamstime.com/illustration/accountant.html

Such as Combinatorial Explosion: ➔ Knapsack or Traveling Salesman Problems (TSP) grow exponentially with problem size.

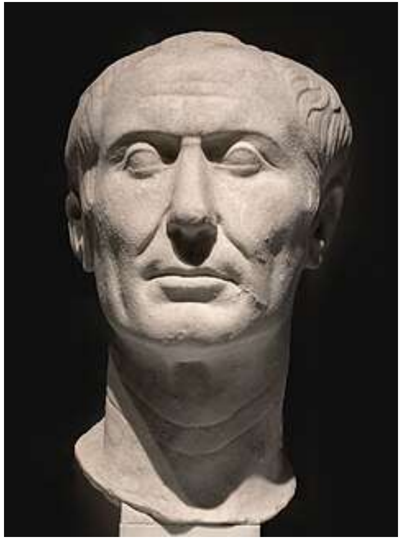
Coupling between Constraints: when the constraints are tightly coupled or interdependent, small changes in one variable can drastically affect the rest of the solution ➔ Network flow problems with constrained multiple edges.

Symmetry when variables or sets of variables can be swapped – causes the solution space to have redundant, equivalent solution ➔ Facility location problems where multiple facilities are identical.

... and more

An illustrative algorithm for integer programming: the branch-and-bound technique:

"The basic concept underlying the branch-and-bound technique is to divide and conquer" (Hillier, p. 502)



Caesar's '*Divide et impera*')…

Julius Caesar
(100-44 BC)

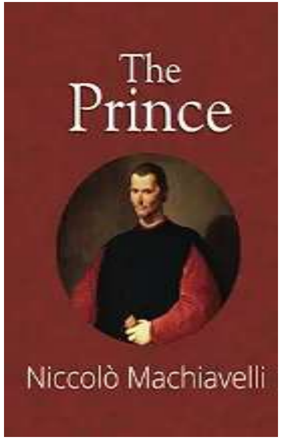
Source: Wikipedia commons

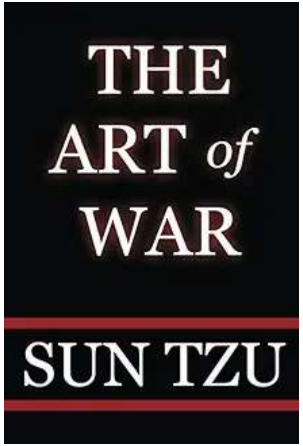
… says Macchiavelli

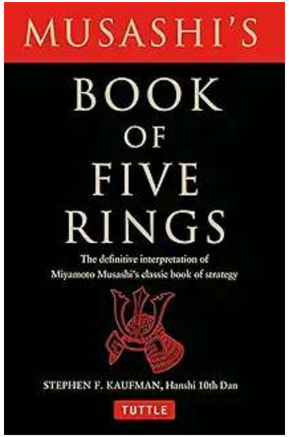
Niccolò Machiavelli
(1469-1527)

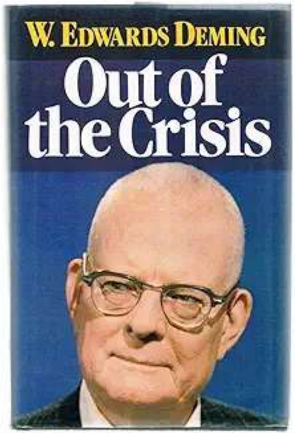
Source: Wikipedia commons

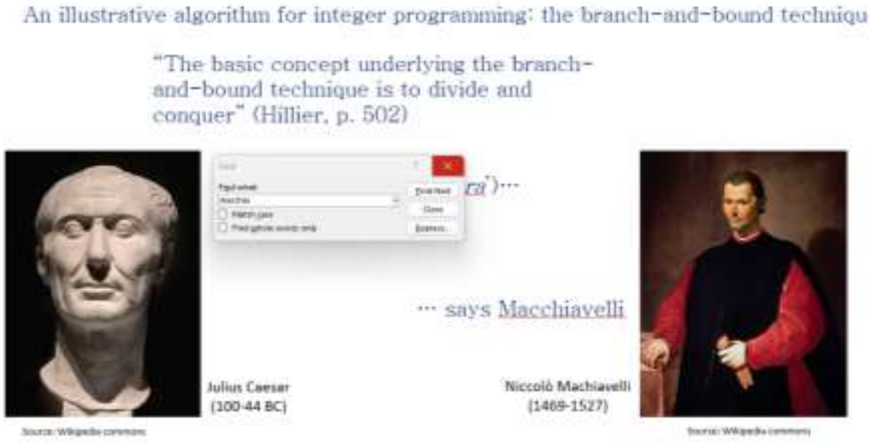
Digression: are these books a ‘must-red’ for your MBA?






MUSASHI'S
BOOK OF FIVE RINGS
The definitive interpretation of Miyamoto Musashi's classic book of strategy
STEPHEN F. KAUFMAN, Hanshi 10th Dan
TUTTLE





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… says Macchiavelli

Julius Caesar
(100-44 BC)

Source: Wikipedia commons

Niccolò Machiavelli
(1469-1527)

Source: Wikipedia commons

Understanding variation using statistical methods, theory of knowledge, psychology, cultural shift and active leadership to break down barriers between departments … ➔ total quality management, Six sigma …

Back to out prototype example: building or not building?

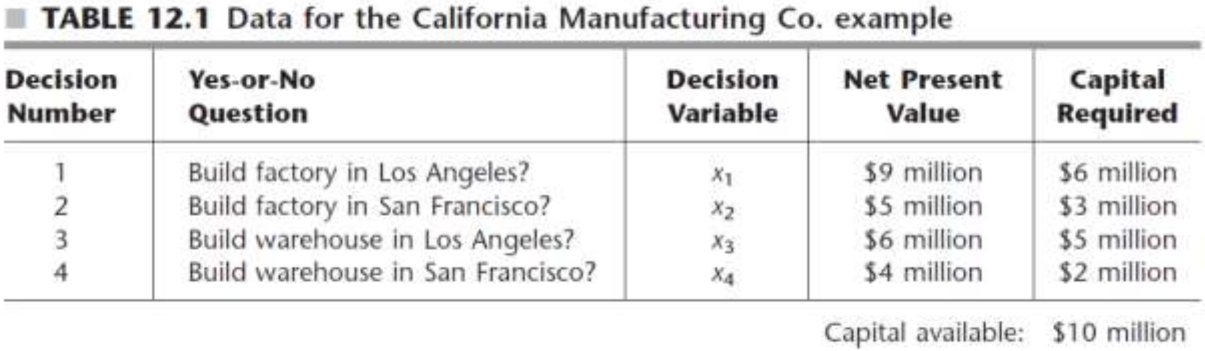
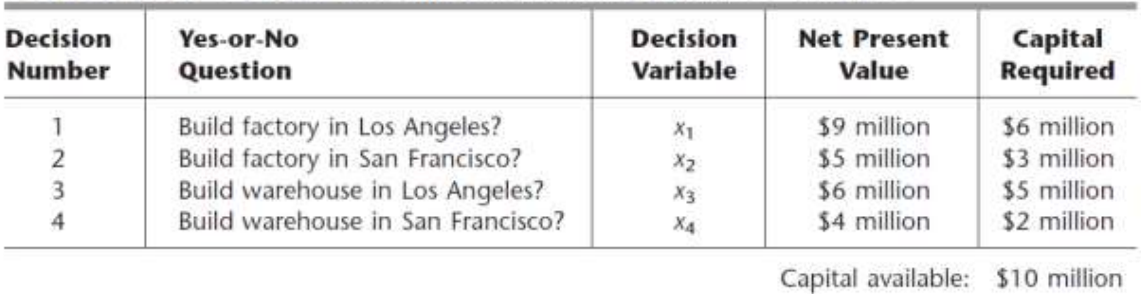


TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

The choice is if building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building **at most one** new warehouse, but the choice of location is restricted to a city where a new factory is being built.

TABLE 12.1 Data for the California Manufacturing Co. example



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4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$

Subject to:

$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$

$-x_1 + x_3 \leq 0$

$-x_2 + x_4 \leq 0$

$x_3 + x_4 \leq 1$

and

x_j binary for $j = 1,2,3,4$

If we apply LP relaxation replacing x_j binary for $j = 1,2,3,4$ with

$x_j \geq 0$ for $j = 1,2,3,4$

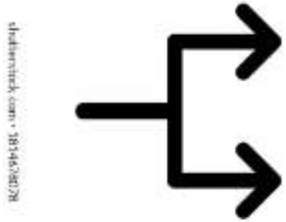
We obtain $x_1, x_2, x_3, x_4 = \left(\frac{5}{6}, 1,0,1\right)$ with $Z = 16.5$; is this a feasible solution?

We **round this to 16** and keep it as an upper bound for the IP problem. Why?

Because all coefficients in the objective function are integer, so all integer solutions must have an integer value for *Z*.

One method to solve IP problems: the branch-and-bound technique

- Branching (split the problem in two branches)
- Bounding (seek for a local optima for Z)
- Fathoming (Resolving the branching at fathomed the node)



Source: https://www.123rf.com/

what are other words for fathomed?
pinpoint, appreciate, plumb, comprehend, unravel, grasp, penetrate, divine, figure out, realized

Source: https://thesaurus.plus/synonyms/fathomed

- Branching (split the problem in two branches), rewriting

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

$x_1 = 0$

$x_1 = 1$

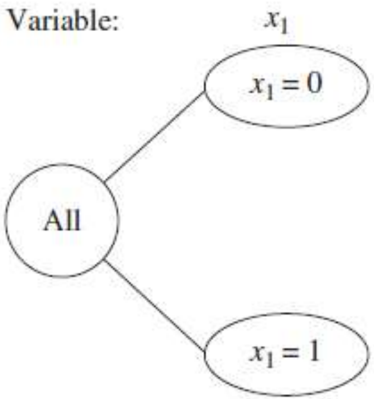
Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \leq 10$
$x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

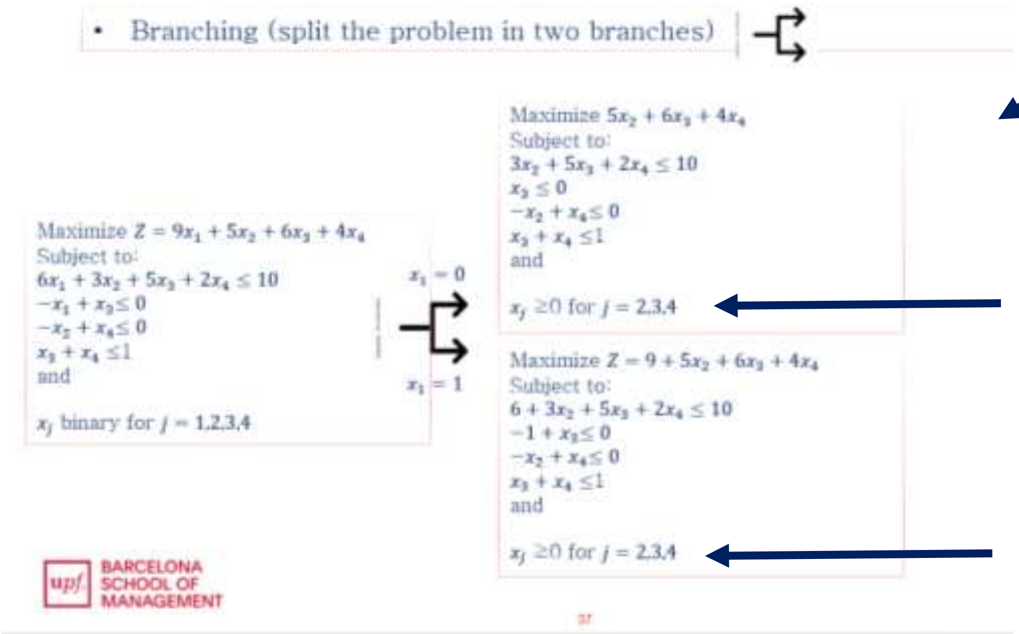
$x_j \geq 0$ for $j = 2,3,4$

(x_1 now disappears)



- Branching (split the problem in two branches)

We are splitting following the order of the variables, i.e. here starting by x_1.



- Branching (split the problem in two branches)

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

$x_1 = 0$

Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \leq 10$
$x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

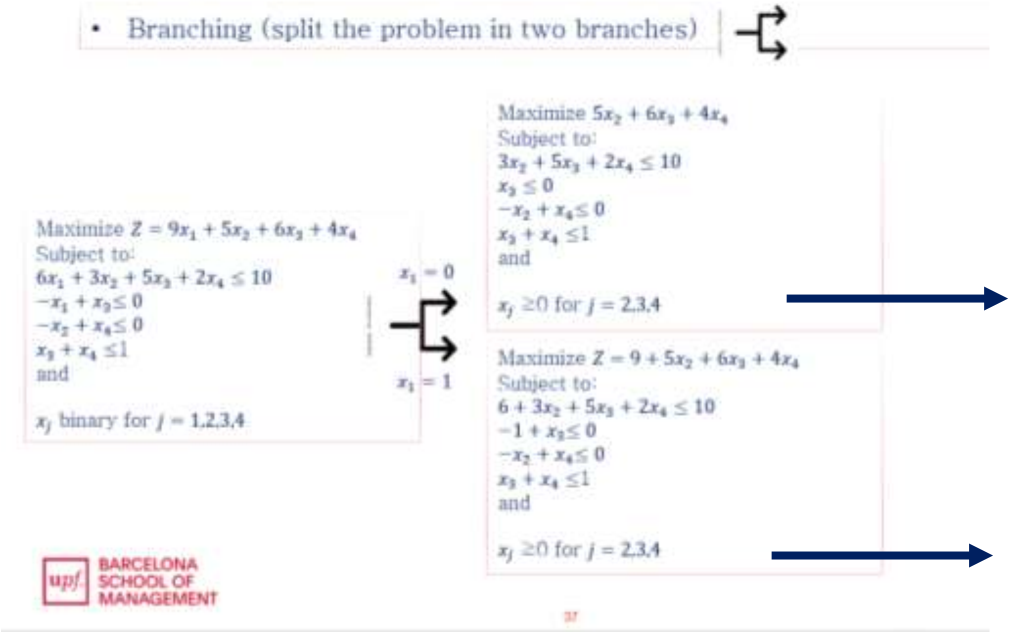
$x_1 = 1$

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

The two subproblems are treated as linear instead of integer

- Bounding (seek for local optima for Z)



- Branching (split the problem in two branches)

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10$
$-x_1 + x_3 \le 0$
$-x_2 + x_4 \le 0$
$x_3 + x_4 \le 1$
and

x_j binary for $j = 1,2,3,4$

$x_1 = 0$

$x_1 = 1$

Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \le 10$
$x_3 \le 0$
$-x_2 + x_4 \le 0$
$x_3 + x_4 \le 1$
and

$x_j \ge 0$ for $j = 2,3,4$

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \le 10$
$-1 + x_3 \le 0$
$-x_2 + x_4 \le 0$
$x_3 + x_4 \le 1$
and

$x_j \ge 0$ for $j = 2,3,4$

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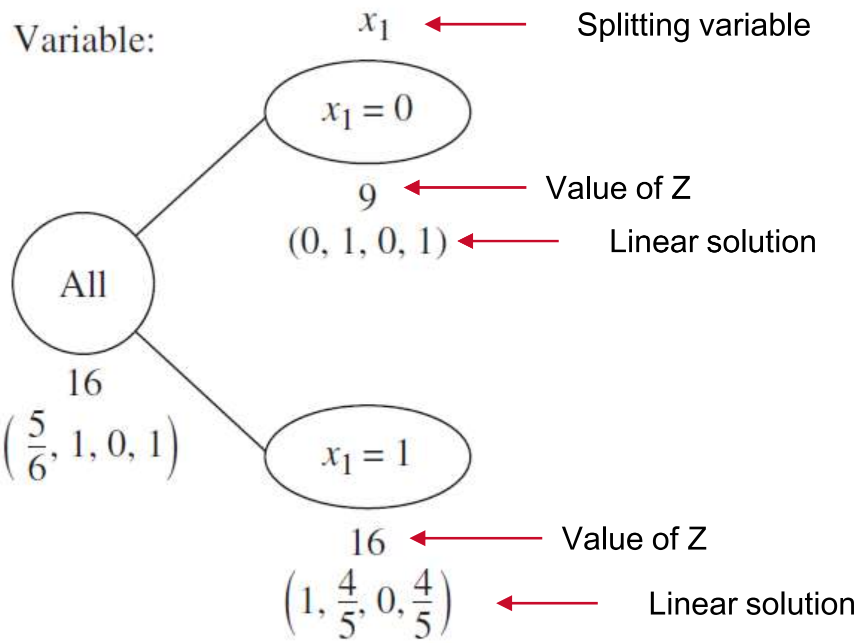
Linear programming applied to these solutions yields

$x_1, x_2, x_3, x_4 = (0,1,0,1)$ with $Z = 9$

$x_1, x_2, x_3, x_4 = \left(1, \frac{4}{5}, 0, \frac{4}{5}\right)$ with $Z = 16.5$

This is where we are at the end of the first bounding step:



Variable: x_1 ← Splitting variable

$x_1 = 0$

9 ← Value of Z

(0, 1, 0, 1) ← Linear solution

All

16

$\left(\frac{5}{6}, 1, 0, 1\right)$

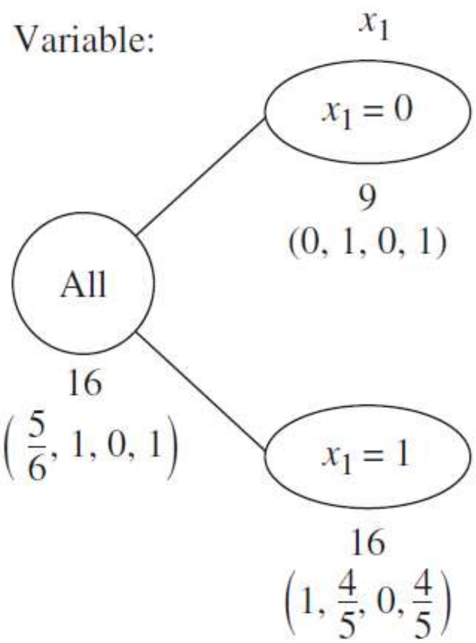
$x_1 = 1$

16 ← Value of Z

$\left(1, \frac{4}{5}, 0, \frac{4}{5}\right)$ ← Linear solution

- Fathoming (Resolving the branching at fathomed the node)

what are other words for fathomed?
pinpoint, appreciate, plumb, comprehend, unravel, grasp, penetrate, divine, figure out, realized
Thesaurus.plus


Variable:
x_1
All
16
$\left(\frac{5}{6}, 1, 0, 1\right)$
$x_1 = 0$
9
(0, 1, 0, 1)
$x_1 = 1$
16
$\left(1, \frac{4}{5}, 0, \frac{4}{5}\right)$

Since it is optimal it does not pay to search for other solutions in this branch

This solution is made of integers! It is **hence optimal for the subproblem with $x_1 = 0$.** We call this now **the incumbent optimum** $Z^* = 9$ and say that the branch $x_1 = 0$ is fathomed; in the following we can get rid of all branches whose $Z \leq Z^* = 9$

This cannot be fathomed; we need to continue

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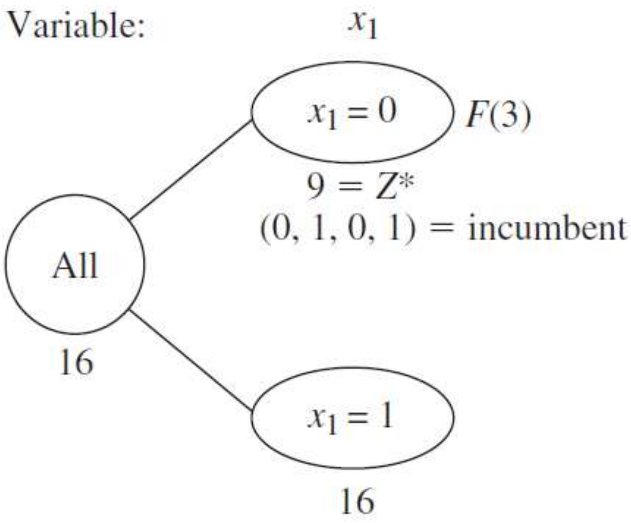
- Fathoming (Resolving the branching at fathomed the node)

what are other words for fathomed?
pinpoint, appreciate, plumb, comprehend, unravel, grasp, penetrate, divine, figure out, realized
Thesaurus.plus



In fact, there are 3 ways of fathoming:

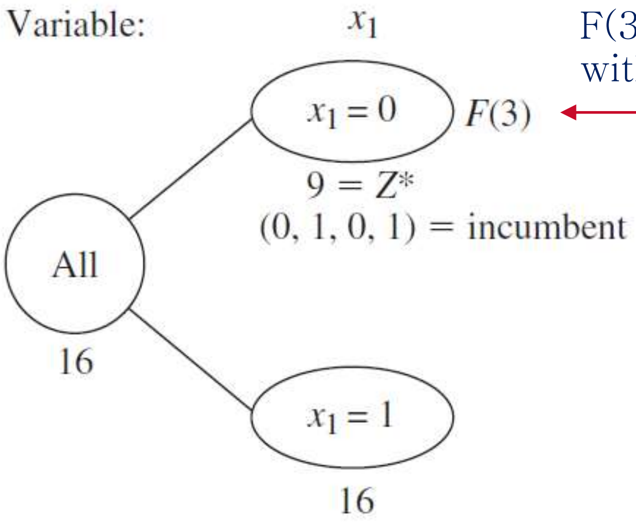
Test 1: Its bound by being $\leq Z^*$

Test 2: Its LP relaxation has no feasible solutions

Test 3: The optimal solution for its LP relaxation is integer.

- Fathoming (Resolving the branching at fathomed the node)

what are other words for fathomed?
pinpoint, appreciate, plumb, comprehend, unravel, grasp, penetrate, divine, figure out, realized
Thesaurus.plus



F(3)=fathomed with test 3

In fact, there are 3 ways of fathoming:

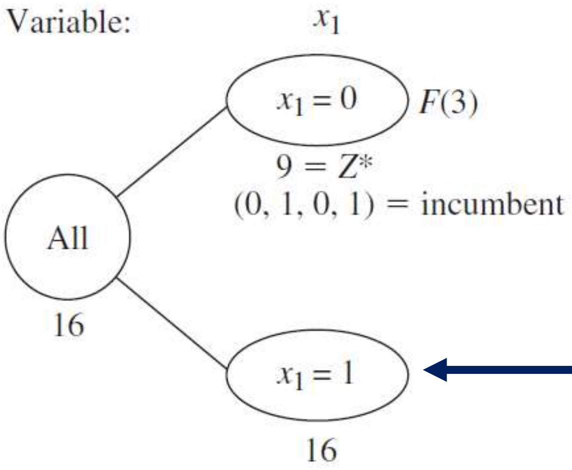
Test 1: Its bound $\leq Z^*$

Test 2: Its LP relaxation has no feasible solutions

Test 3: The optimal solution for its LP relaxation is integer

If a solution is better than the incumbent, it becomes the new incumbent Z^*, and test 1 is reapplied to all **previous unfathomed** subproblems using this new larger Z^*

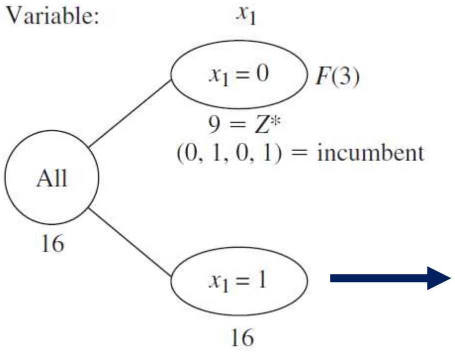
- Continuing the example branching on variable x_2



We now branch the $x_1 = 1$ problem by branching x_2 between 0 and 1

- Continuing the example

(x_2 now disappears)



$x_2 = 0, x_1 = 1$

Maximize $Z = 9 + 6x_3 + 4x_4$

Subject to

Subject to:

$5x_3 + 2x_4 \leq 4$

$x_3 \leq 1$

$x_4 \leq 0$

$x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 3,4$

$x_2 = 1, x_1 = 1$

Maximize $Z = 9 + 5 + 6x_3 + 4x_4$

Subject to:

$5x_3 + 2x_4 \leq 1$

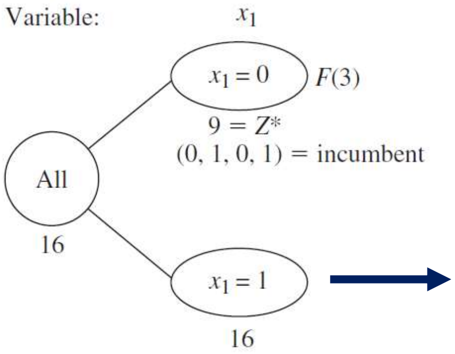
$x_3 \leq 1$

$x_4 \leq 1$

$x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 3,4$

- Continuing the example



$x_2 = 0, x_1 = 1$
Maximize $Z = 9 + 6x_3 + 4x_4$
Subject to
Subject to:
$5x_3 + 2x_4 \leq 4$
$x_3 \leq 1$
$x_4 \leq 0$
$x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 3,4$

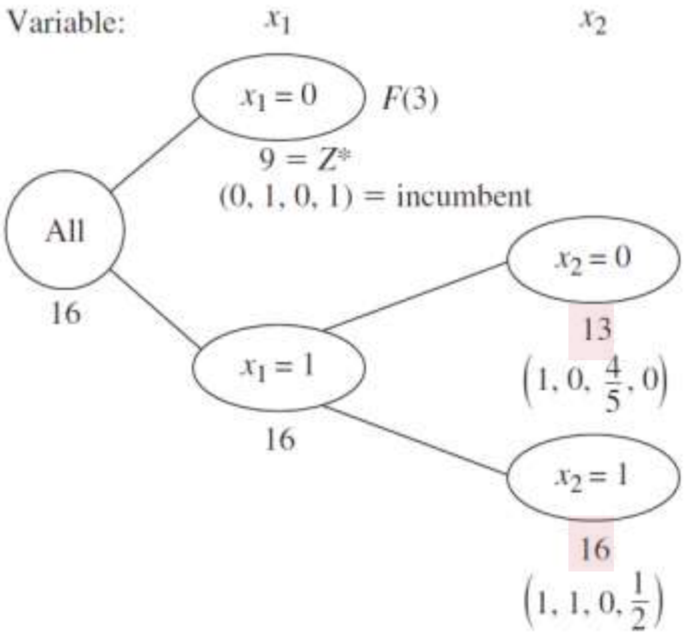
Linear programming applied to these solutions yields

$$x_1, x_2, x_3, x_4 = \left(1,0,\frac{4}{5},0\right) \text{ with } Z = 13.8$$

$x_2 = 1, x_1 = 1$
Maximize $Z = 9 + 5 + 6x_3 + 4x_4$
Subject to:
$5x_3 + 2x_4 \leq 1$
$x_3 \leq 1$
$x_4 \leq 1$
$x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 3,4$

$$x_1, x_2, x_3, x_4 = \left(1,1,0,\frac{1}{2}\right) \text{ with } Z = 16$$

- Continuing the example


Variable:
x_1
x_2
All
16
$x_1 = 0$
$F(3)$
$9 = Z^*$
$(0, 1, 0, 1)$ = incumbent
$x_1 = 1$
16
$x_2 = 0$
13
$\left(1, 0, \frac{4}{5}, 0\right)$
$x_2 = 1$
16
$\left(1, 1, 0, \frac{1}{2}\right)$

Why do I say so when
they are not integer ?

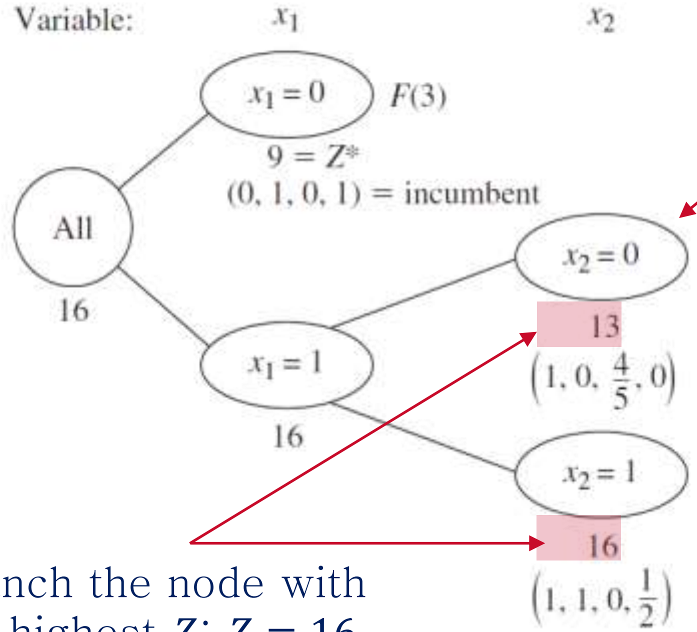
This is where we are now; no problem has been bound or fathomed at this step because

Test 1: Its bound $\le Z^*$ No, both 13 and 16 > 9

Test 2: Its LP relaxation has no feasible solutions False; both are feasible

Test 3: The optimal solution for its LP relaxation is integer No for both

- Continuing the example


Variable:
x_1
x_2
$x_1 = 0$
$F(3)$
$9 = Z^*$
$(0, 1, 0, 1)$ = incumbent
All
16
$x_1 = 1$
16
$x_2 = 0$
13
$\left(1, 0, \frac{4}{5}, 0\right)$
$x_2 = 1$
16
$\left(1, 1, 0, \frac{1}{2}\right)$

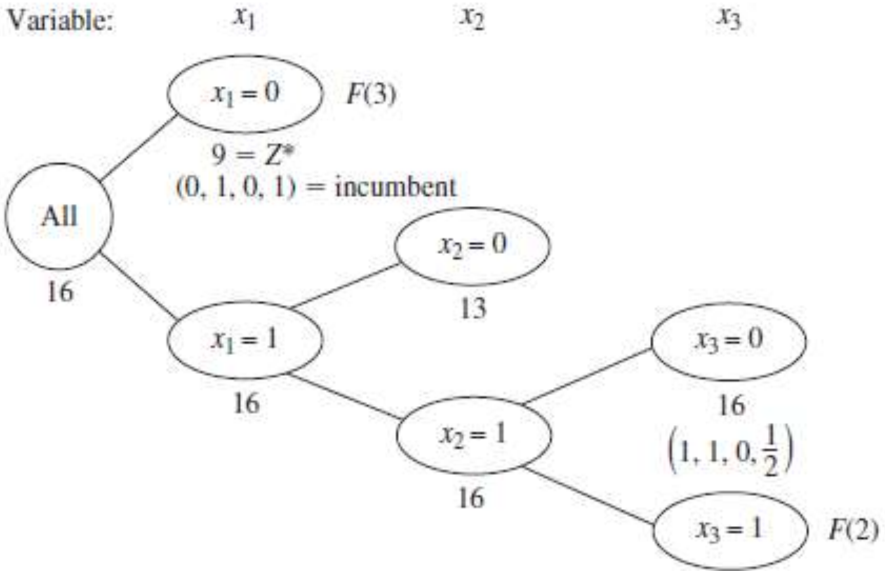
… but keep in mind that this is still open (unfathomed)

Since the problem $x_2 = 1$ has the larger Z we branch this solution splitting on x_3

Branch the node with the highest Z: $Z = 16$

- Continuing the example branching x_3 ; note how both Z and the constraints change to adopt to the new values

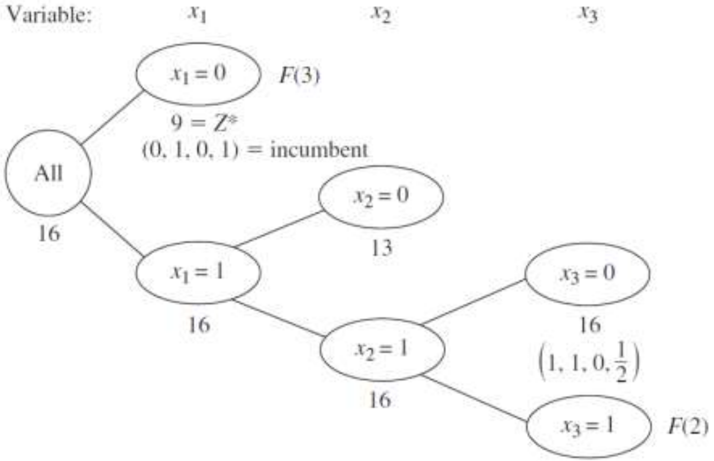
(x_3 now disappears)



$x_3 = 0$, $x_1 = 1$, $x_2 = 1$
Maximize $Z = 14 + 4x_4$
Subject to:
$2x_4 \leq 1$ This was $5x_3 + 2x_4 \leq 1$
$x_4 \leq 1$
$x_4 \leq 1$ This was $x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 4$

$x_3 = 1$, $x_1 = 1$, $x_2 = 1$
Maximize $Z = 20 + 4x_4$
Subject to:
$2x_4 \leq -4$ This was $5x_3 + 2x_4 \leq 1$
$x_4 \leq 1$
$x_4 \leq 0$ This was $x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 4$

- Continuing the example



$x_3 = 0, x_1 = 1,\ x_2 = 1$
Maximize $Z = 14 + 4x_4$
Subject to:
$2x_4 \leq 1$
$x_4 \leq 1$
$x_4 \leq 1$
$x_j \geq 0$ for $j = 4$

Linear programming applied to these solutions yields no feasible **integer** solution

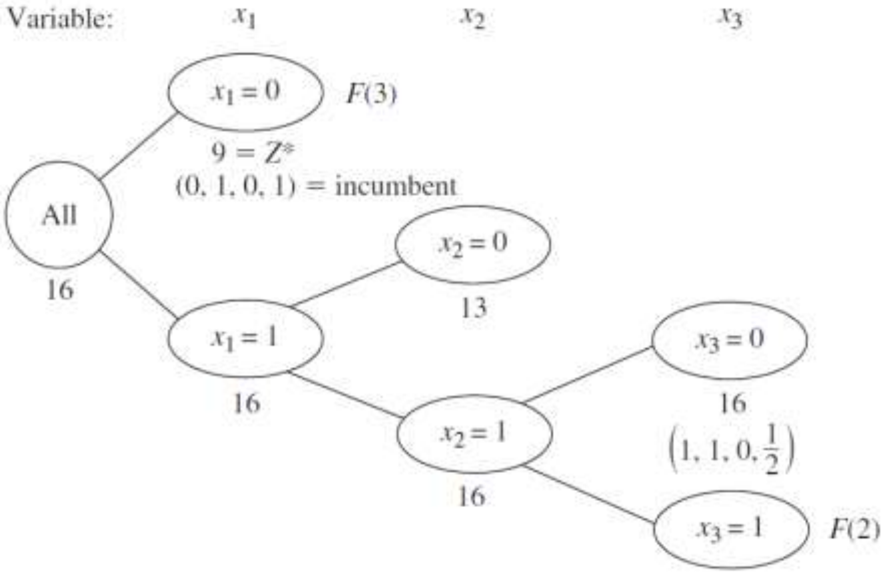
$x_1, x_2, x_3, x_4 = \left(1,1,0,\frac{1}{2}\right)$ with $Z = 16$

$x_3 = 1, x_1 = 1,\ x_2 = 1$
Maximize $Z = 20 + 4x_4$
Subject to:
$2x_4 \leq -4$
$x_4 \leq 1$
$x_4 \leq 0$
$x_j \geq 0$ for $j = 4$

x_1, x_2, x_3, x_4
= no feasible solution

Why?

- Continuing the example



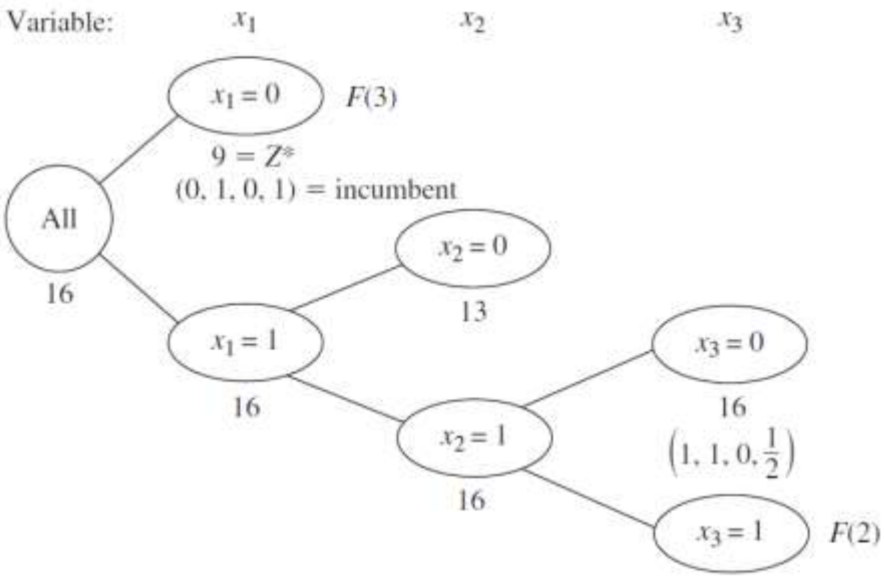
This is where we are now, with one solution fathomed and one open

No test failed

Test 2 failed

what are other words for fathomed?
pinpoint, appreciate, plumb, comprehend, unravel, grasp, penetrate, divine, figure out, realized
Thesaurus.plus

- Continuing the example by branching the last variable x_4



We now branch the problem from $x_3 = 0$, but since only variable x_4 is left fixing it generates directly a solution!

For $x_4 = 0$
$x_1, x_2, x_3, x_4 = (1,1,0,0)$ with $Z = 14$

For $x_4 = 1$
$x_1, x_2, x_3, x_4 = (1,1,0,1)$ unfeasible

We can check this last passage with the original problem

For $x_4 = 0$
$x_1, x_2, x_3, x_4 = (1,1,0,0)$ with $Z = 14$

For $x_4 = 1$
$x_1, x_2, x_3, x_4 = (1,1,0,1)$ unfeasible

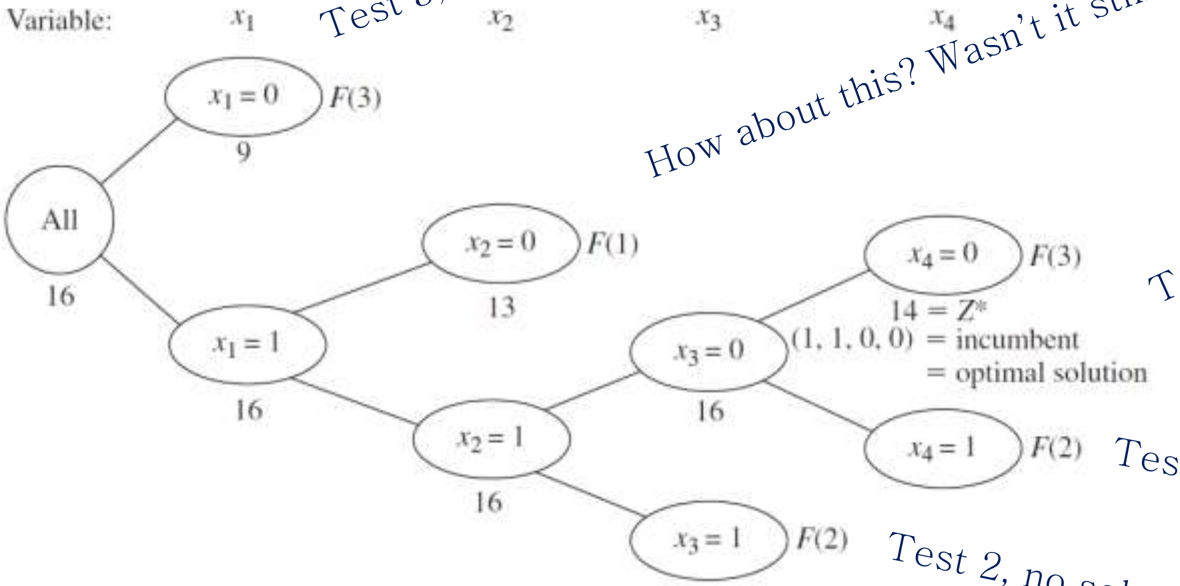
Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

$9 \leq 10$

$11 \leq 10$

- Revising and closing: the solution is $x_1, x_2, x_3, x_4 = (1,1,0,0)$ with $Z = 14$, but ⋯



Test 3, integer solution

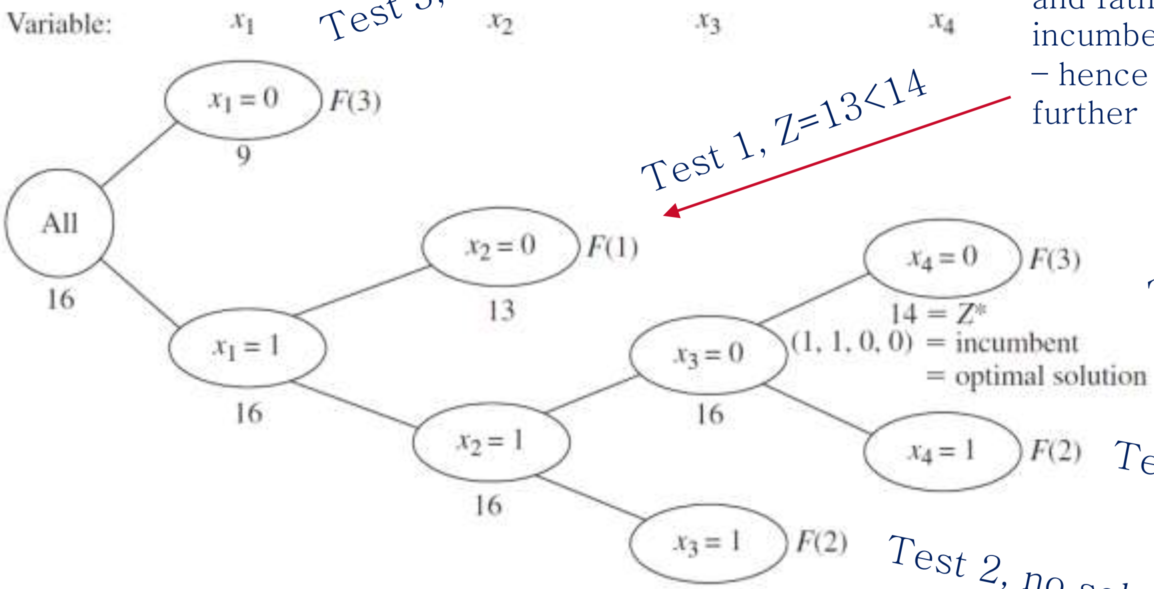
How about this? Wasn't it still open?

Test 3, integer, new incumbent

Test 2, no solution

Test 2, no solution

- Revising and closing: the solution is $x_1, x_2, x_3, x_4 = (1,1,0,0)$ with $Z = 14$



Test 3, integer solution

Test 1, Z=13<14

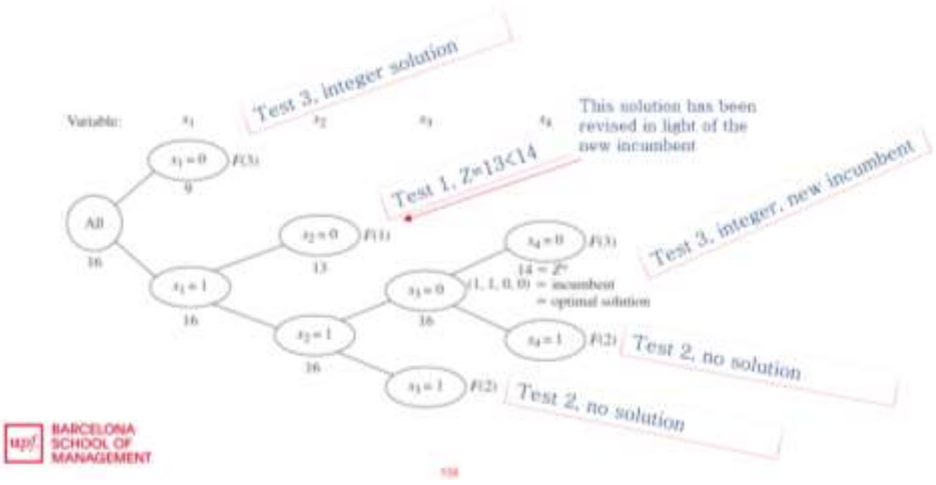
This solution has been revised and fathomed in light of the new incumbent (that has $Z = 14 > 13$) – hence no need to explore further

Test 3, integer, new

nbent

Test 2, no solution

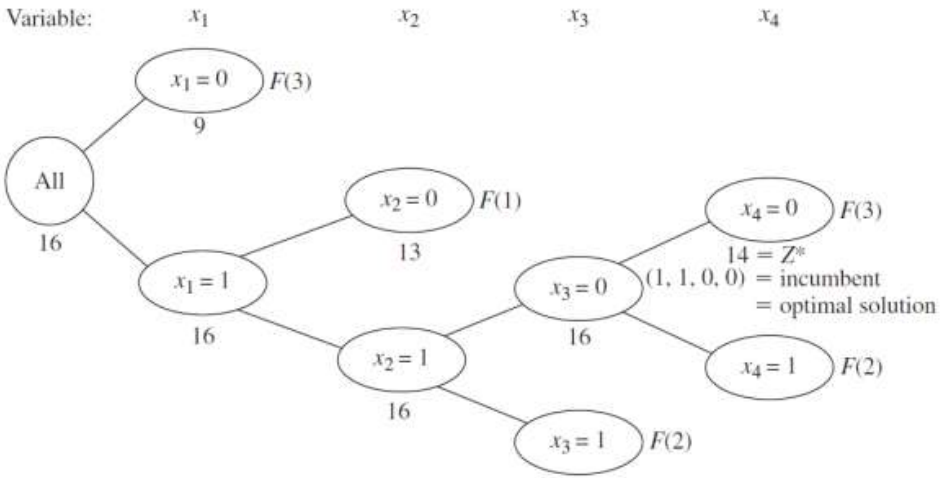
Test 2, no solution



The solution is laborious, Needs book-keeping of how objective and constraints change in the various branches, and repeated recourse to LP, simplex calculations

But this was a very simple problem in 4 variables;

- Would it have been possible to solve this problem ‘brute force’?
- How many leaves would there be in this tree in total?
- Would we need to apply LP?



0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	8
1001	9
1010	10
1011	11
1100	12
1101	13
1110	14
1111	15

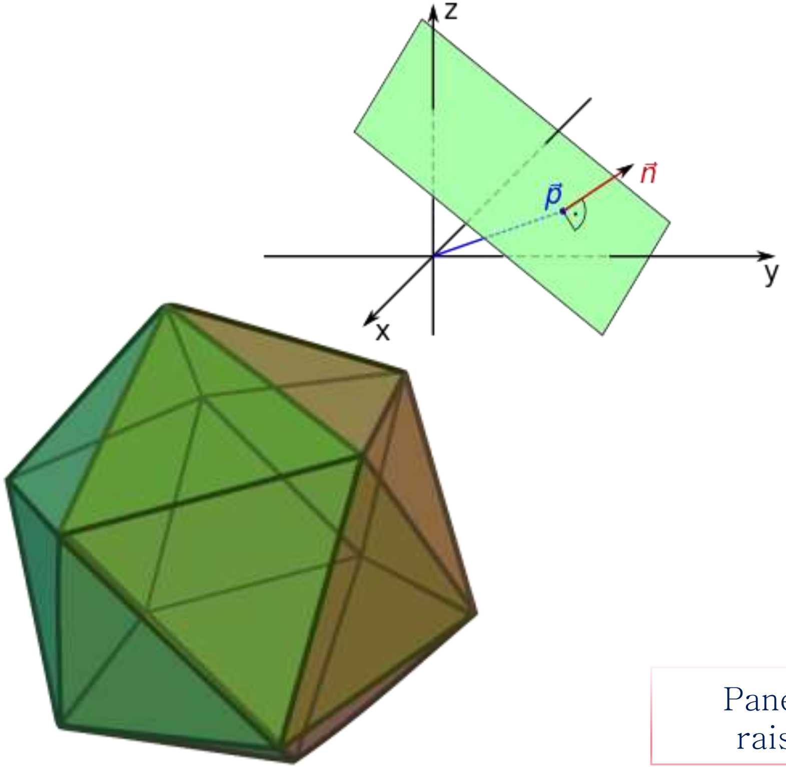
Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

Some take home points

Integer programming and linear programming: LP=convex polyhedron touched by the hyperplane of the objective function; the IP solutions instead are isolated point inside the polyhedron

Find these points may not be easy but the LP solution is an upper bound for the Z of IP



Source (both images): Wikipedia Commons

Panettone with raisins inside

Source: https://leitesculinaria.com/478/recipes-cranberry-pistachio-panettone.html

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16.

Nonlinear Programming

Problem framing and examples. Graphical illustration. Bisection and Newton methods of solution. Metaheuristics. Genetic algorithms. Examples of nonlinear models. Python coding. Exploration versus optimization. Hillier 2014, chapters 13 and 14.

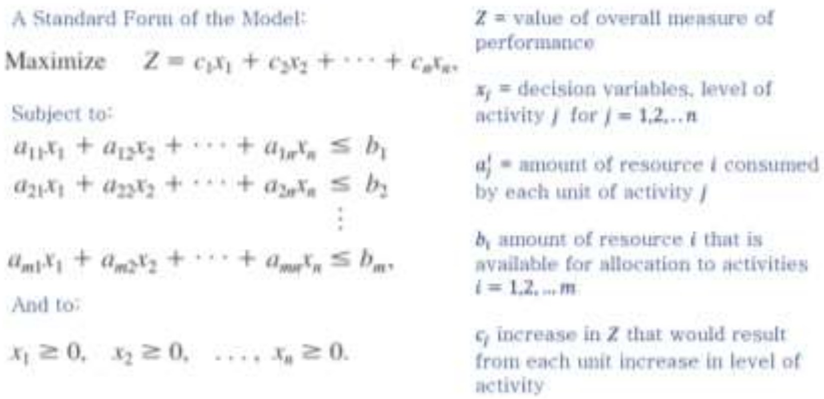
Problem setting

Finding values of $\boldsymbol{x} = (x_1, x_2, \dots x_n)$ as to maximize or minimize a generic function $f(\boldsymbol{x})$ subject to

$$g_i(\boldsymbol{x}) \le b_i \quad i = 1,2, \dots m$$

and

$$\boldsymbol{x} \ge 0$$



A Standard Form of the Model:

Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$,

Subject to:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m, \end{aligned}$$

And to:

$$x_1 \ge 0, \quad x_2 \ge 0, \quad \dots, \quad x_n \ge 0.$$

Z = value of overall measure of performance

x_j = decision variables, level of activity j for $j = 1,2,\dots n$

a_j^i = amount of resource i consumed by each unit of activity j

b_i amount of resource i that is available for allocation to activities $i = 1,2,\dots m$

c_j increase in Z that would result from each unit increase in level of activity

What is the difference from the linear problem?

Problem setting

Finding values of $\boldsymbol{x} = (x_1, x_2, \dots x_n)$ as to maximize or minimize a generic function $f(\boldsymbol{x})$ subject to

← Is this specification necessary?

$g_i(\boldsymbol{x}) \leq b_i \quad i = 1,2, \dots m$

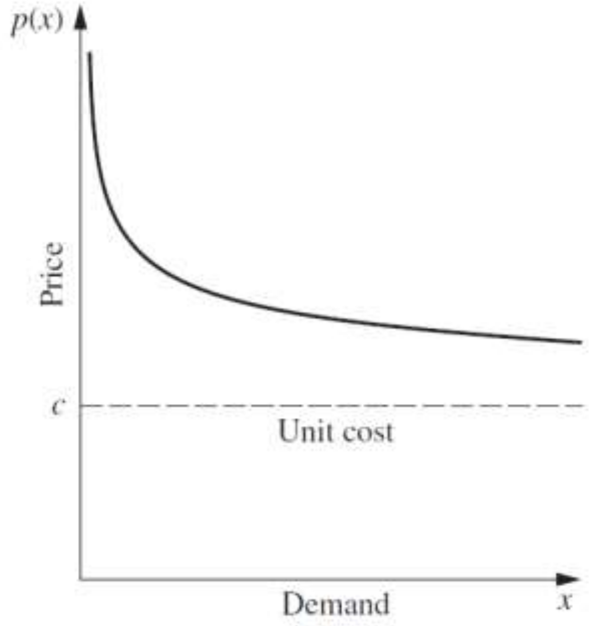
and

$\boldsymbol{x} \geq 0$

Many ways in which a linear problem can become nonlinear

Linear: there is a fixed unit profit associated with each product, so the resulting objective function will be linear

Nonlinear: prices $\boldsymbol{p(x)}$ are subject to elasticity



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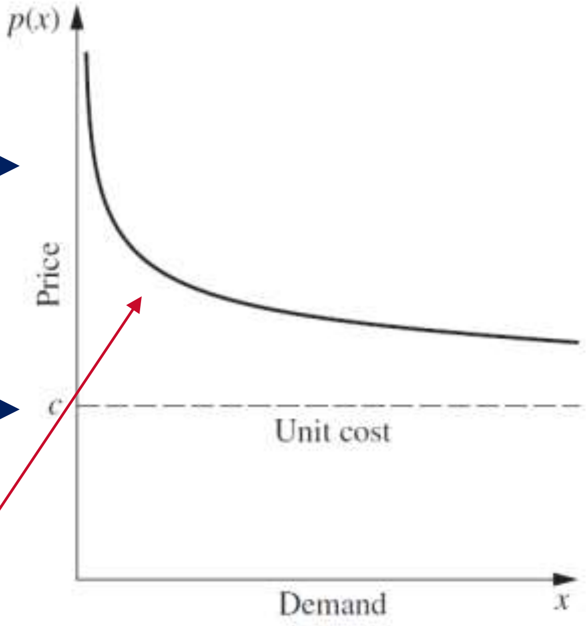
The firm’s profit P from producing and selling x units is given by the nonlinear function

$$P(x) = xp(x) - cx$$



But the higher the price, the less is sold

The production cost remain constant

How should this curve change to make the problem linear?

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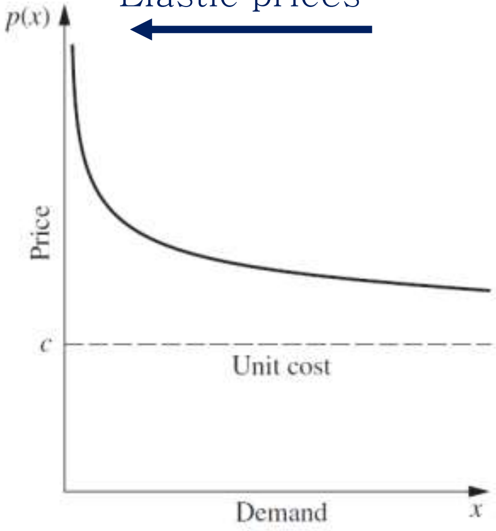
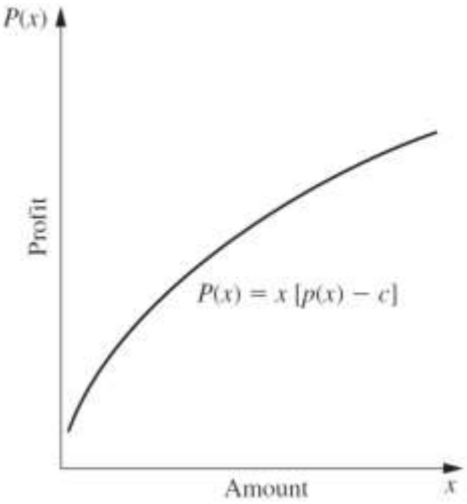
Elastic prices ⋯



⋯ Lead to non linear profits



The firm's profit P from producing and selling x units is given by the nonlinear function

$$P(x) = xp(x) - cx$$

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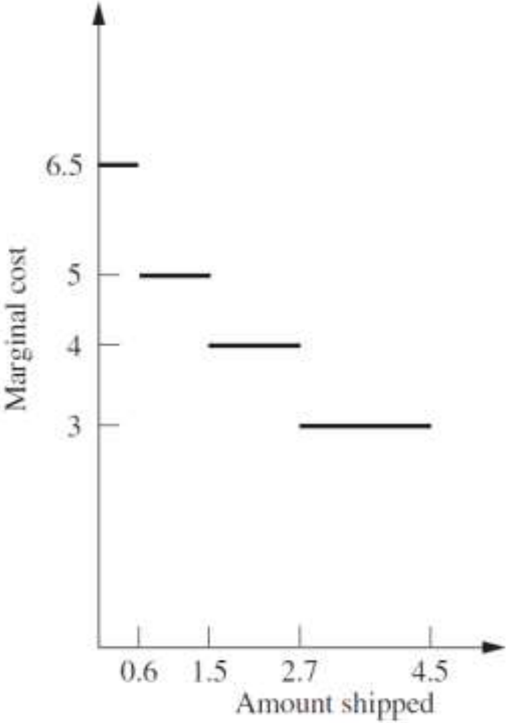
But the production cost may as well be a non linear function, e.g. in the case of increasing or diminishing returns

$$P(x) = xp(x) - cx$$

Source:https://www.bbcgoodfood.com/howto/guide/health-benefits-bananas

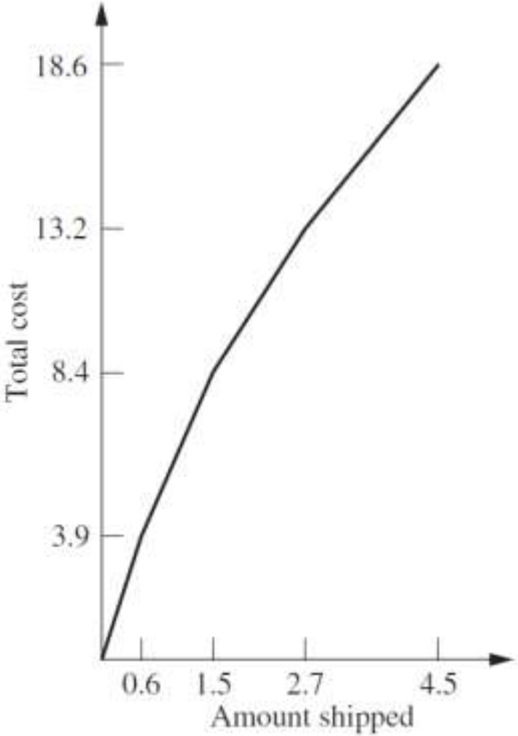
Source:https://www.mercedes-benz.es/passengercars/models/coupe/new/cle.html

In transportation problems the shipping costs may decrease with the volume shipped



Decreasing unitary cost …

… Piecewise total cost



In portfolio modelling the decision variable could be the number of shares of a given stock $x_j, i = 1,2, \ldots n$ to be included

Assume known the mean return of stock μ_j and its variance σ_{jj}. Also assume that σ_{jj} is a proxy of the risk for that stock.

But the fluctuations of the stock are not independent, se we also need to know the covariances $\sigma_{ij}, i \neq j$, $i, j = 1,2 \ldots n$

Return from the entire portfolio $R(\boldsymbol{x}) = \sum_j^n \mu_j x_j$

Variance $V(\boldsymbol{x})$ of the total return $V(\boldsymbol{x}) = \sum_{j=1}^n \sum_{i=1}^n \sigma_{ij} x_i x_j$

Source: https://www.britannica.com/money/topic/stock-exchange-finance

So the nonlinear problem is
Minimize

$$V(\boldsymbol{x}) = \sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_{ij} x_i x_j$$

But is this correct?

⋯ or should it be

$$V(\boldsymbol{x}) = \sum_{j=1}^{n}\sum_{i=1}^{j} \sigma_{ij} x_i x_j$$

$$V(\boldsymbol{x}) = \sum_{j=1}^{n}\sum_{i=1}^{j} \sigma_{ij} x_i x_j$$

Subject to

$\sum_j^n \mu_j x_j \geq \boldsymbol{L}$ where $\boldsymbol{L}$ is the minimum profit desired

$\sum_j^n P_j x_j \leq \boldsymbol{B}$ where P_j is the cost of stock j and $\boldsymbol{B}$ is the budget available for the portfolio

$x_j \geq 0, i = 1,2, \ldots n$
(Hiller, p. 551)

Source: https://www.britannica.com/money/topic/stock-exchange-finance

So the nonlinear problem is
Minimize

$$V(\boldsymbol{x}) = \sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_{ij} x_i x_j$$

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$\sum_j^n P_j x_j \leq \boldsymbol{B}$ where P_j is the cost of stock j and $\boldsymbol{B}$ is the budget available for the portfolio

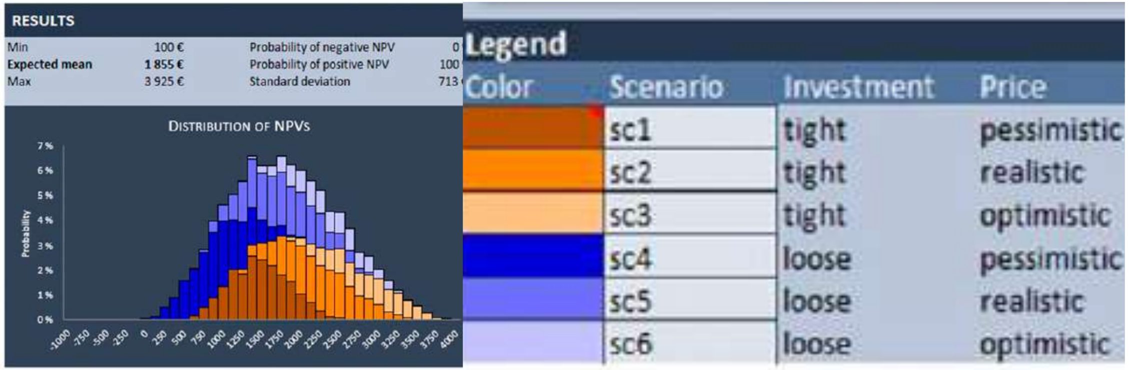
$x_j \geq 0, i = 1,2, \dots n$

There is a trade off between $\boldsymbol{L}$ and $V(\boldsymbol{x})$

(higher profit associated to higher risk)

So that this problem is solved for a range of values of $\boldsymbol{L}$ and comparing for each $\boldsymbol{L}$ the associated $R(\boldsymbol{x}) = \sum_j^n \mu_j x_j$ and $V(\boldsymbol{x})$

"Therefore, rather than stopping with one choice of *L*, it is common to use a *parametric* (nonlinear) programming approach to generate the optimal solution as a function of *L* over a wide range of values of *L*. The next step is to examine the values of *R*(**x**) and *V*(**x**) for these solutions that are optimal for some value of *L* and then to choose the solution that seems to give the best trade-off between these two quantities." (Hillier, p. 552)



RESULTS

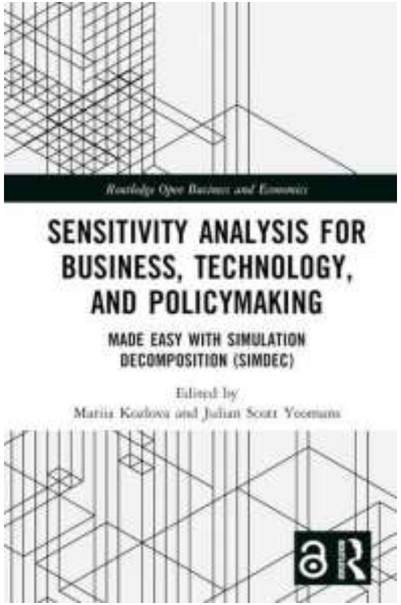
Min	100 €	Probability of negative NPV	0
Expected mean	**1 855 €**	Probability of positive NPV	100
Max	3 925 €	Standard deviation	713

Legend

Color	Scenario	Investment	Price
🟫	sc1	tight	pessimistic
🟧	sc2	tight	realistic
🟨	sc3	tight	optimistic
🟦	sc4	loose	pessimistic
🟪	sc5	loose	realistic
⬜	sc6	loose	optimistic

From Lesson #3

Agree: better to look at the entire output distribution of the return $R(\boldsymbol{x}) = \sum_j^n \mu_j x_j$, including its tails, and have a sensitivity analysis to go with, as to learn where the worst outcomes are coming from



Routledge Open Business and Economics

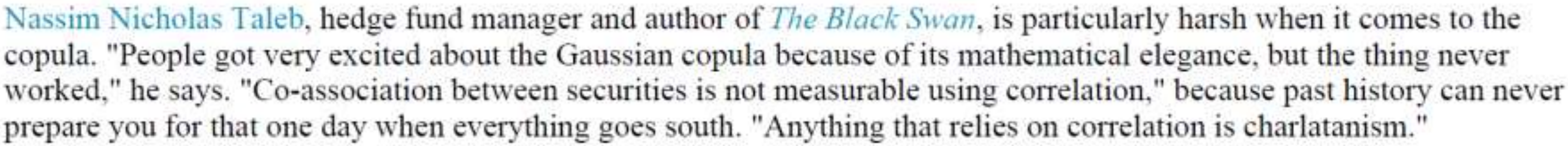
SENSITIVITY ANALYSIS FOR BUSINESS, TECHNOLOGY, AND POLICYMAKING

MADE EASY WITH SIMULATION DECOMPOSITION (SIMDEC)

Edited by
Mariia Kozlova and Julian Scott Yeomans

"Therefore, rather than stopping with one choice of *L*, it is common to use a *parametric* (nonlinear) programming approach to generate the optimal solution as a function of *L* over a wide range of values of *L*. The next step is to examine the values of *R*(**x**) and *V*(**x**) for these solutions that are optimal for some value of *L* and then to choose the solution that seems to give the best trade-off between these two quantities." (Hillier, p. 552)

The estimation of the σ_{ij} is a delicate matter; the case of the subprime mortgage crisis



Nassim Nicholas Taleb, hedge fund manager and author of *The Black Swan*, is particularly harsh when it comes to the copula. "People got very excited about the Gaussian copula because of its mathematical elegance, but the thing never worked," he says. "Co-association between securities is not measurable using correlation," because past history can never prepare you for that one day when everything goes south. "Anything that relies on correlation is charlatanism."

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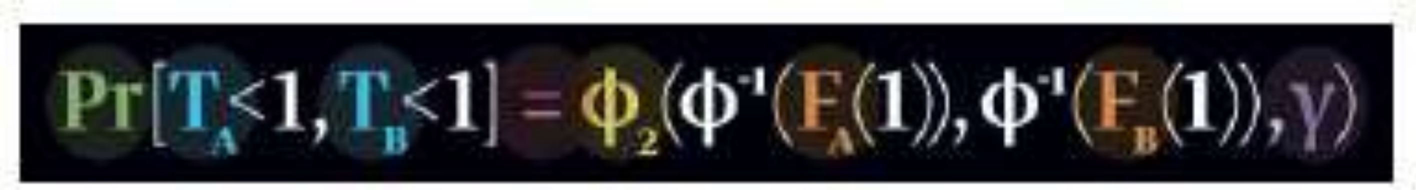
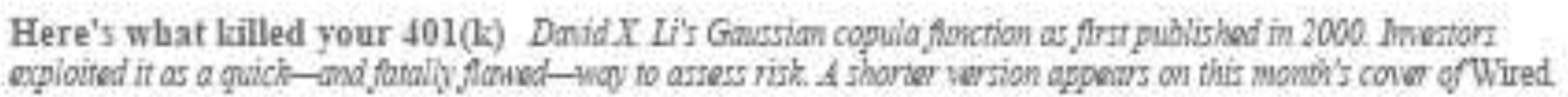
WIRED MAGAZINE: 17.03

Recipe for Disaster: The Formula That Killed Wall Street

By Felix Salmon 02.23.09



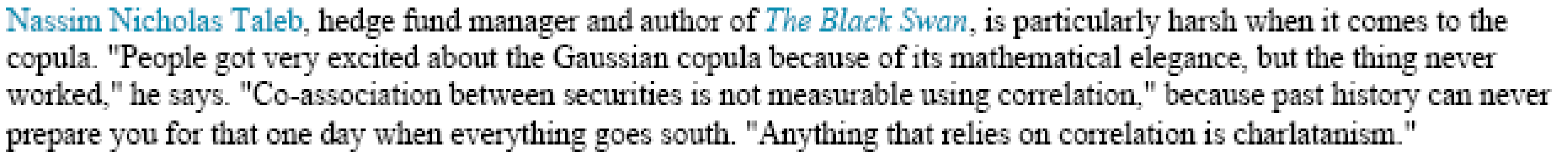
$$\Pr[T_A<1,T_B<1]=\phi_2(\phi^{-1}(F_A(1)),\phi^{-1}(F_B(1)),\gamma)$$



Here's what killed your 401(k) *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick—and fatally flawed—way to assess risk. A shorter version appears on this month's cover of Wired.*

Here is what killed your 401(k)…

Li’s Gaussian copula function …



Nassim Nicholas Taleb, hedge fund manager and author of *The Black Swan*, is particularly harsh when it comes to the copula. "People got very excited about the Gaussian copula because of its mathematical elegance, but the thing never worked," he says. "Co-association between securities is not measurable using correlation," because past history can never prepare you for that one day when everything goes south. "Anything that relies on correlation is charlatanism."

Felix Salmon, Wired, February 2009

Source: https://www.wired.com/2009/02/wp-quant/

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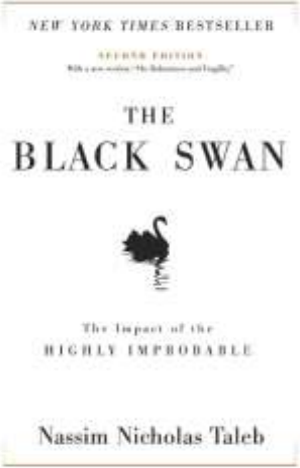
WIRED

Financial econometrics has always been risky

Source https://www.fynsa.com/en/newsletter/

Rober Merton, Myron Scholes and Fisher Black

The story and more here



NEW YORK TIMES BESTSELLER

THE BLACK SWAN

The Impact of the HIGHLY IMPROBABLE

Nassim Nicholas Taleb

Scholes and Merton were awarded the 1973 (so-called) Nobel Prize in Economic Sciences (Black had passed away)

Scholes and Merton created the Long-Term Capital Management fund in 1994

In 1998 the fund had to be bailed out by the Feds

What killed LTCP
- Extreme market volatility,
- Correlated risk across seemingly unrelated assets,
- Inadequate attention to liquidity risks,
- Assumptions of normal market behaviour, and
- Excessive leverage.

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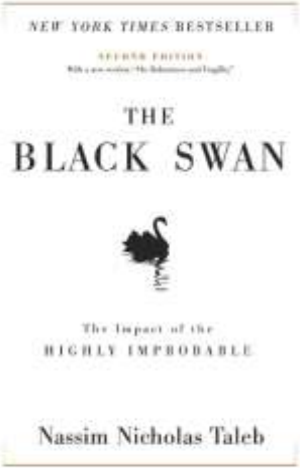
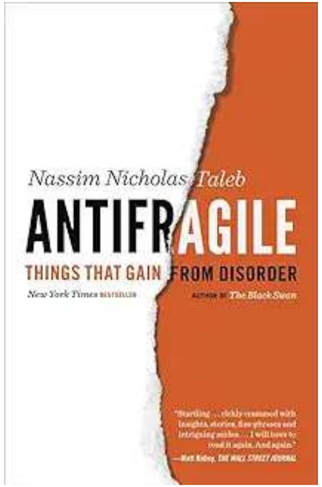
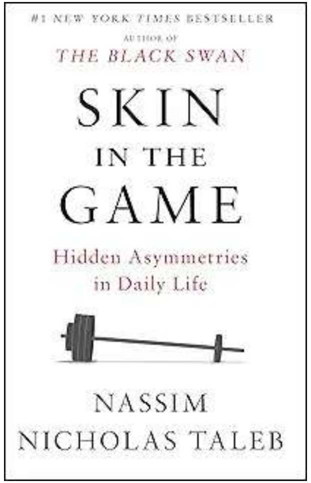
Since we mentioned him …


NEW YORK TIMES BESTSELLER
THE BLACK SWAN
The Impact of the HIGHLY IMPROBABLE
Nassim Nicholas Taleb

2010



2014



2018

Nassim Nicholas Taleb

Source: Wikipedia Commons

Back to Hiller: an old example made nonlinear

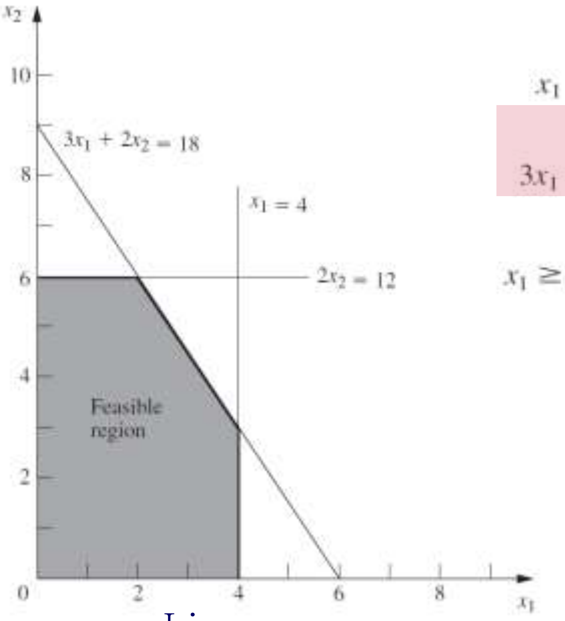
$$\text{Maximize} \quad Z = 3x_1 + 5x_2,$$

$$x_1 \leq 4$$

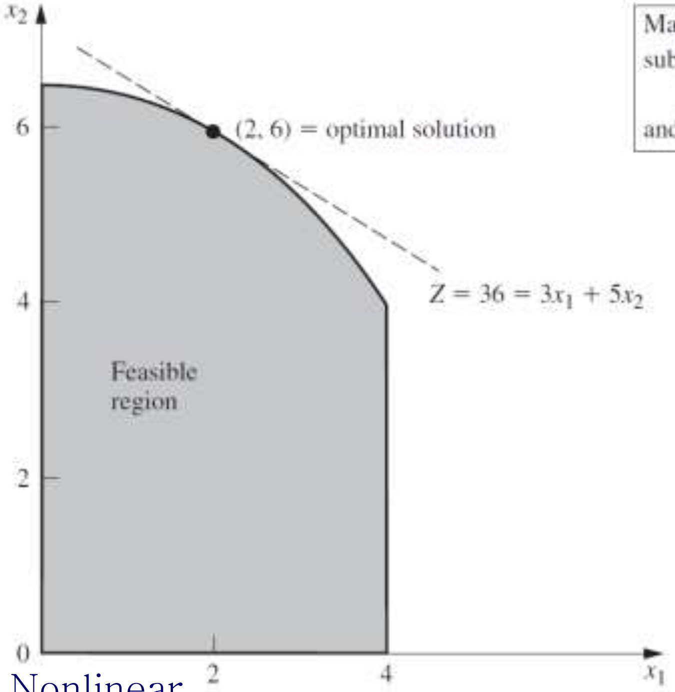
$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

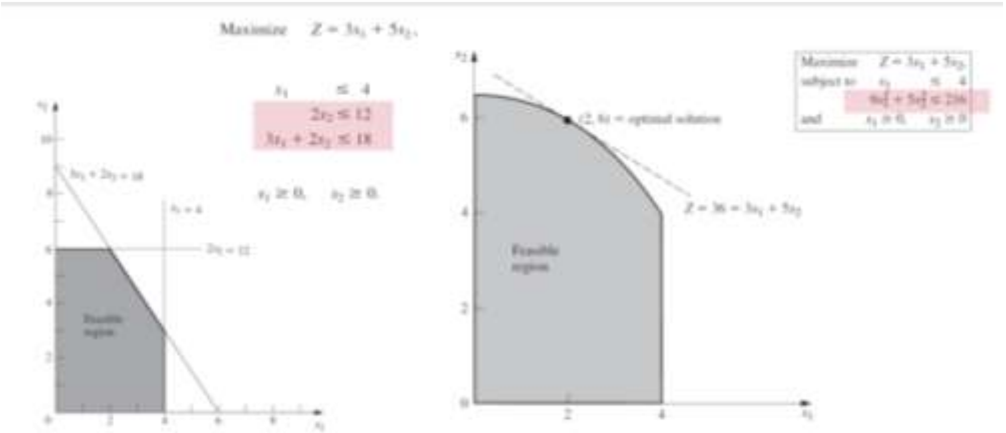


Linear



Nonlinear

$$\begin{array}{lll} \text{Maximize} & Z = 3x_1 + 5x_2, & \\ \text{subject to} & x_1 \leq 4 & \\ & 9x_1^2 + 5x_2^2 \leq 216 & \\ \text{and} & x_1 \geq 0, & x_2 \geq 0 \end{array}$$



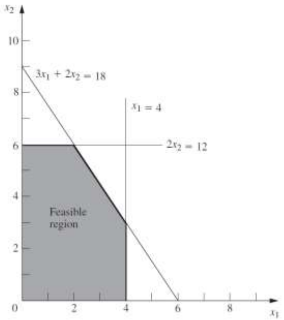
The solution still happens to be on the boundary of the feasible region, but there are no longer the corner points feasible (CPF) to help us

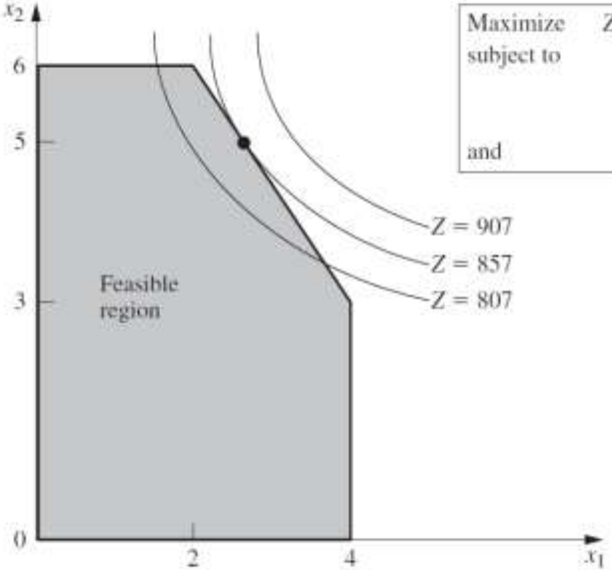
If we now maintain the old constraints of the linear problem i.e.

$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

But change the objective function to a nonlinear form

$$Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2$$





Maximize $Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2$,
subject to

$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}$$

and $x_1 \geq 0, \quad x_2 \geq 0$

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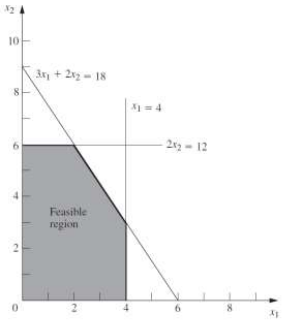
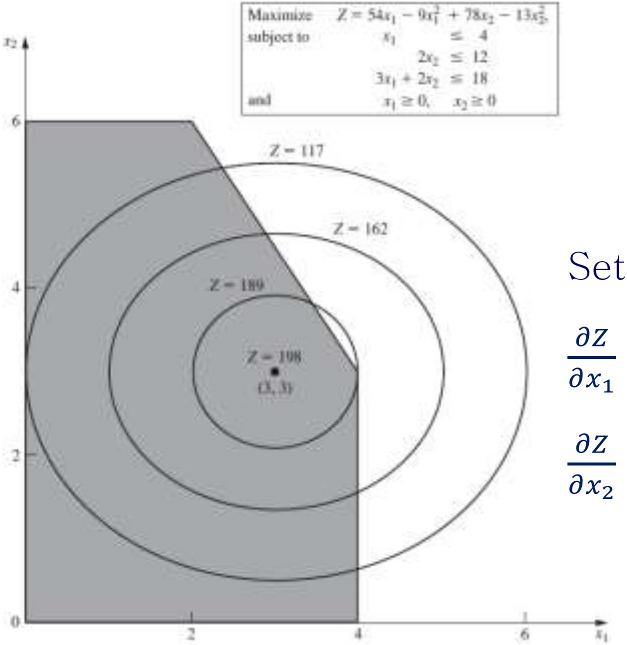
If we now maintain the old constraints of the linear problem i.e.

$$
\begin{aligned}
x_1 &\le 4\\
2x_2 &\le 12\\
3x_1 + 2x_2 &\le 18\\
x_1 &\ge 0\\
x_2 &\ge 0
\end{aligned}
$$

…or to another nonlinear form

$$Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2$$





$$
\begin{array}{lll}
\text{Maximize} & Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2, & \\
\text{subject to} & x_1 \le 4 & \\
 & 2x_2 \le 12 & \\
 & 3x_1 + 2x_2 \le 18 & \\
\text{and} & x_1 \ge 0, \quad x_2 \ge 0 &
\end{array}
$$

The point $x_1, x_2 = 3{,}3$ is a maximum.

Differentiate with respect to x_1, x_2

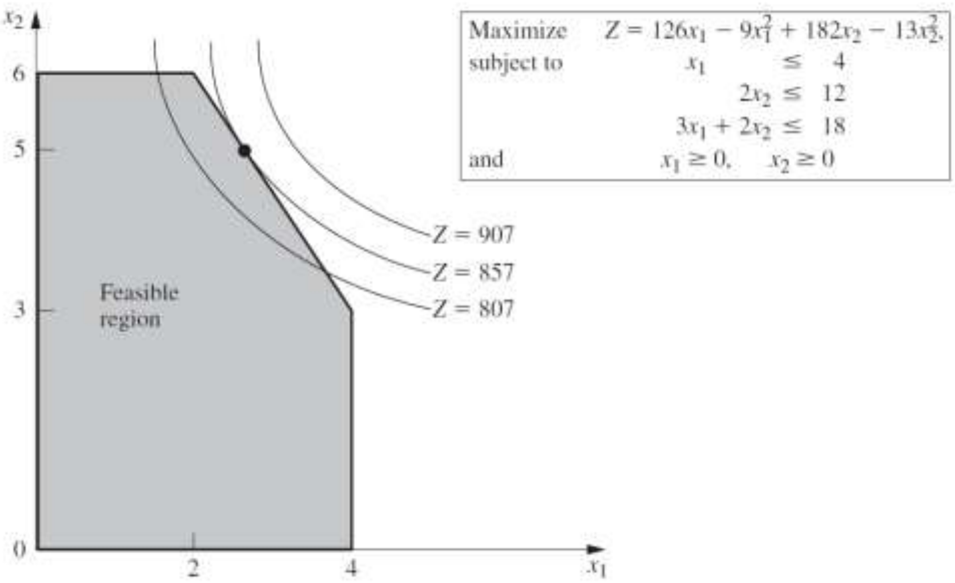
$$Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2$$

$$\frac{\partial Z}{\partial x_1} = 54 - 18x_1$$
$$\frac{\partial Z}{\partial x_2} = 78 - 26x_2$$

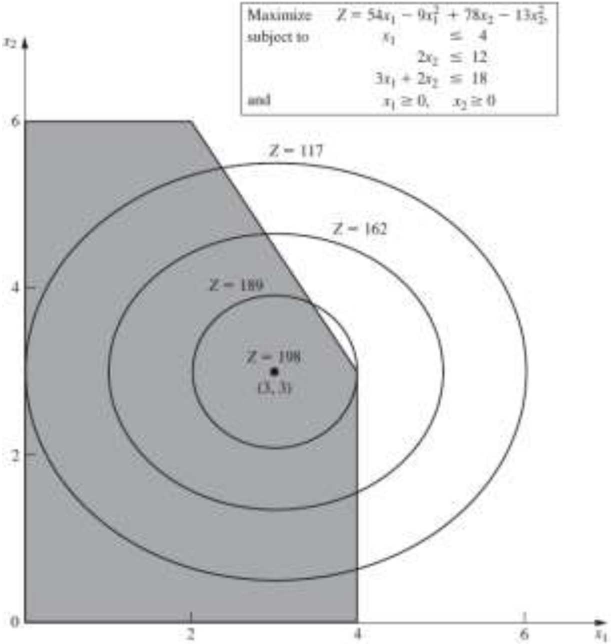
Setting both derivatives to zero

$\frac{\partial Z}{\partial x_1} = 0$ gives $x_1 = 3$

$\frac{\partial Z}{\partial x_2} = 0$ gives $x_2 = 3$



$$\begin{array}{llrcl}
\text{Maximize} & Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2, & & & \\
\text{subject to} & & x_1 & \le & 4 \\
 & & 2x_2 & \le & 12 \\
 & & 3x_1 + 2x_2 & \le & 18 \\
\text{and} & & x_1 \ge 0, & & x_2 \ge 0
\end{array}$$

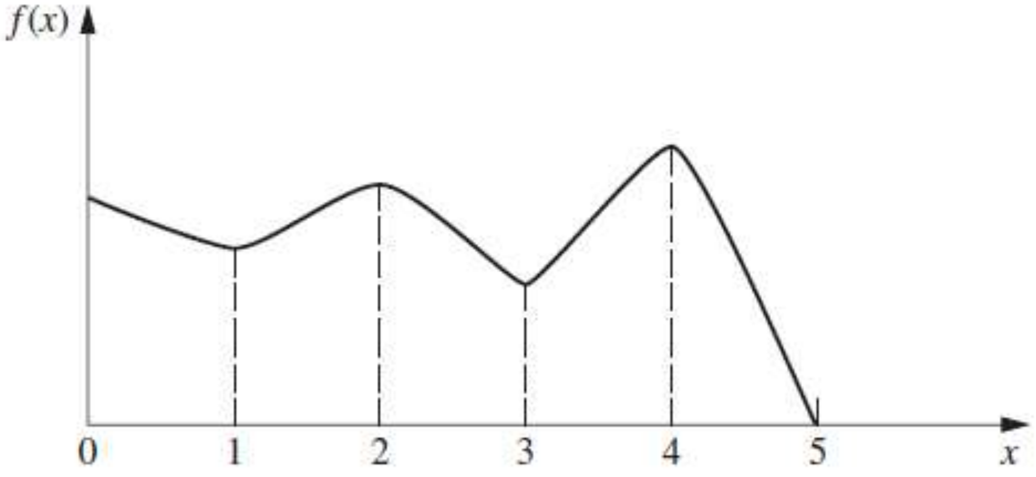


$$\begin{array}{llrcl}
\text{Maximize} & Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2, & & & \\
\text{subject to} & & x_1 & \le & 4 \\
 & & 2x_2 & \le & 12 \\
 & & 3x_1 + 2x_2 & \le & 18 \\
\text{and} & & x_1 \ge 0, & & x_2 \ge 0
\end{array}$$

The solution can be anywhere in the feasible region, no longer just on its frontier

Other complications of the nonlinear problem: there can be more maxima

FIGURE 13.8
A function with several local maxima ($x = 0, 2, 4$), but only $x = 4$ is a global maximum.



In order to have just one maximum the objective function must be **concave** over the entire dominion of the search. Similarly for a **convex** function there is guarantee of just one minimum

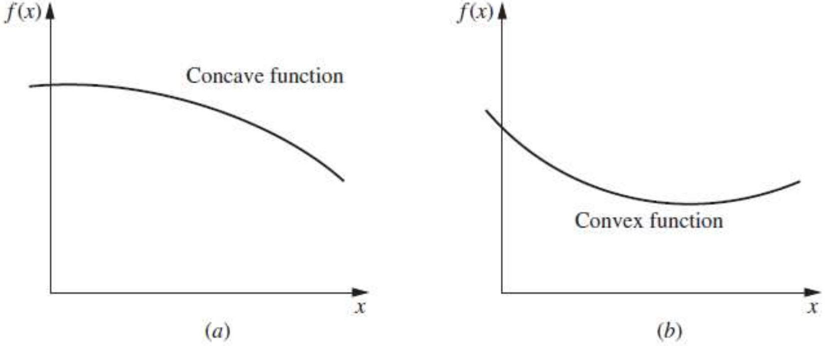


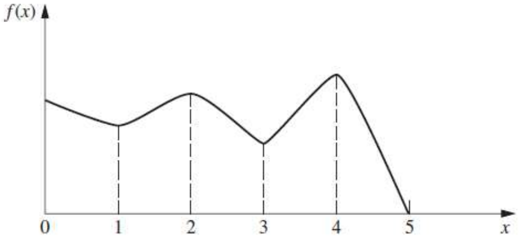
FIGURE 13.9
Examples of (*a*) a concave function and (*b*) a convex function.

$$\frac{\partial^2 f(x)}{\partial x^2} \leq 0$$

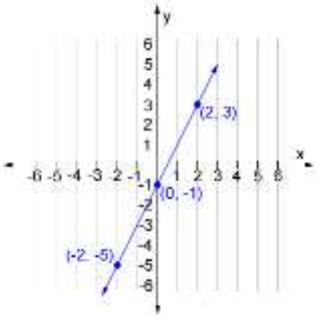
concave

$$\frac{\partial^2 f(x)}{\partial x^2} \geq 0$$

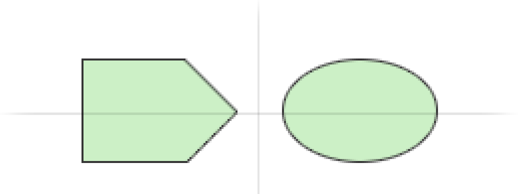
convex



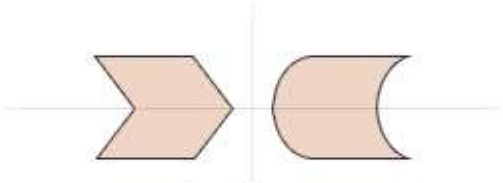
Concave or convex?



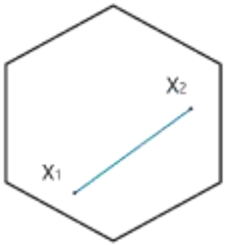
Concave or convex?



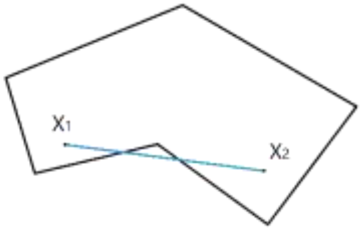
Convex



Non-Convex



Convex set

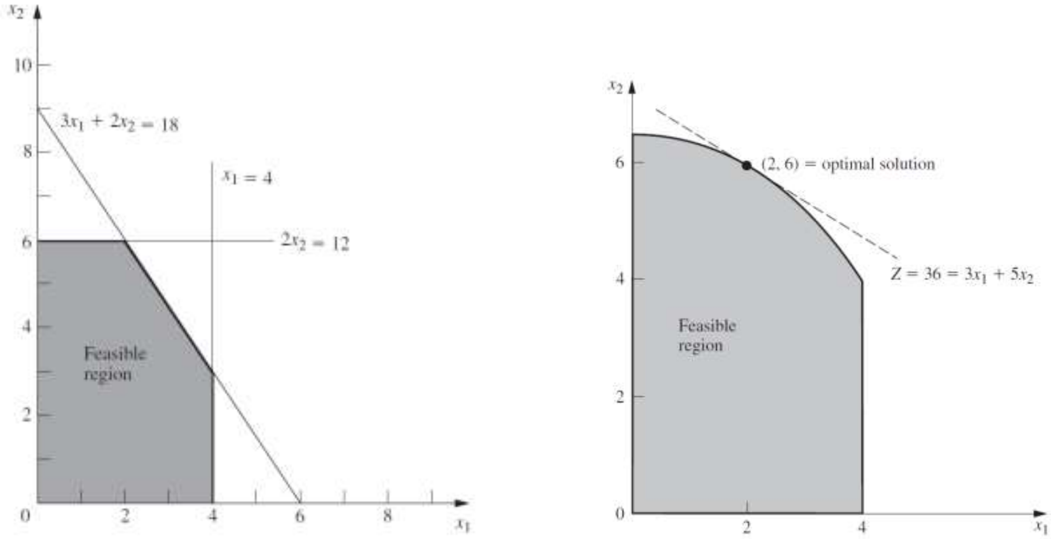


Non-convex set

In the case of constraints the existence of a global maximum is conditioned by the shape of the feasible region. If the feasible region is a **convex set**, then the global maximum is ensured.

"A convex set is simply a set of points such that, for each pair of points in the collection, the entire line segment joining these two points is also in the collection"

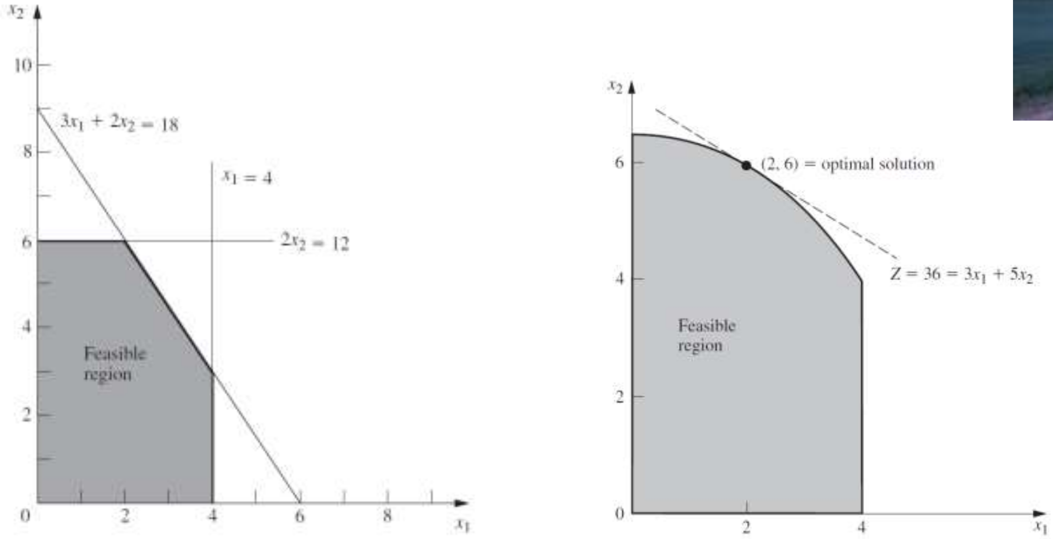
Note: in linear programming the feasible region is always a convex set



The feasible region is a convex set in both these cases

= you can reach each point in the set from each other point in the set by **walking in a straight line**

Source:
https://mwpetersonlaw.com/



The feasible region is a convex set in both these cases

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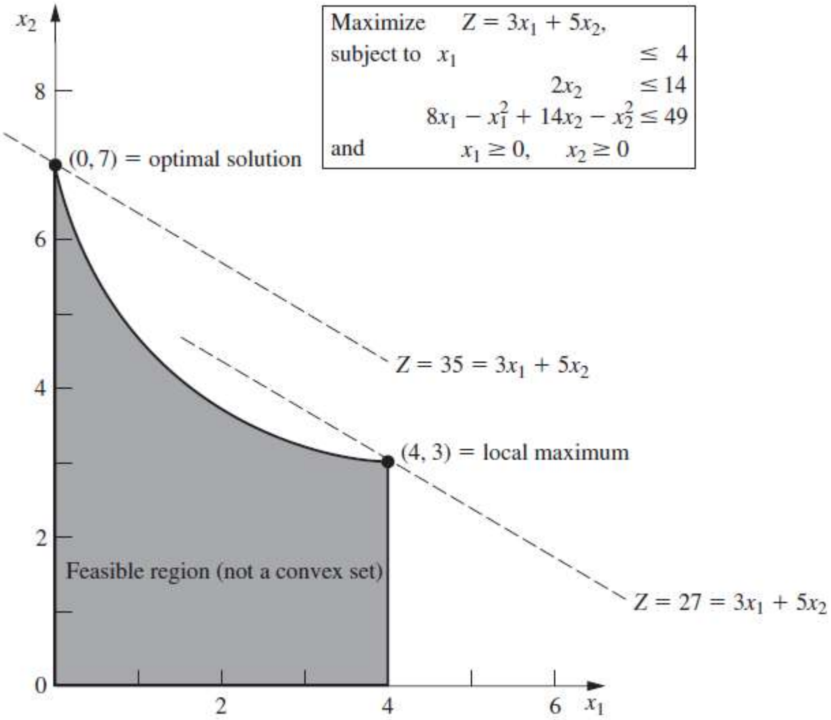
This is not the case here



$$\begin{array}{ll} \text{Maximize} & Z = 3x_1 + 5x_2, \\ \text{subject to} & x_1 \le 4 \\ & 2x_2 \le 14 \\ & 8x_1 - x_1^2 + 14x_2 - x_2^2 \le 49 \\ \text{and} & x_1 \ge 0, \quad x_2 \ge 0 \end{array}$$

a

d

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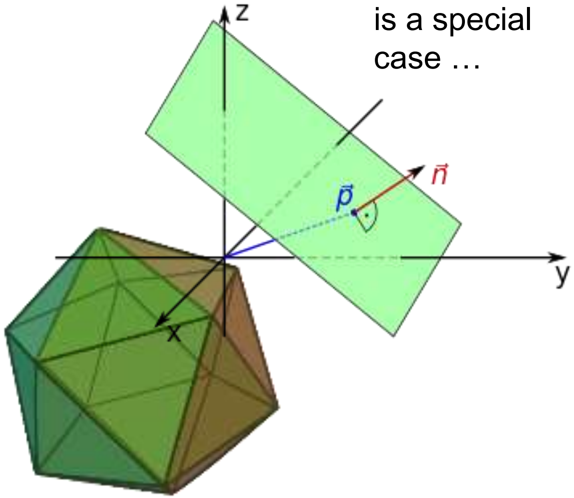
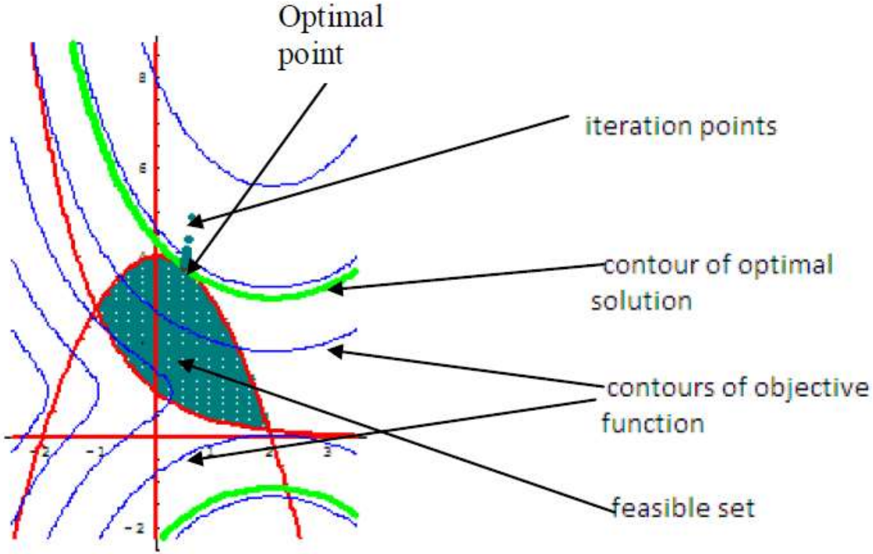
A nonlinear programming problem with constraints $g_i(\boldsymbol{x}) \leq b_i \quad i = 1,2, \dots m$ has a **global maximum** when the objective $f(x)$ is a concave function and all the $g_i(\boldsymbol{x})$ are convex

➔ This is known as convex programming problem

In fact this is a special case …



… of that



Source: https://www.researchgate.net/publication/315450072_Penalty_function_methods_using_matrix_laboratory_MATLAB/

The minimum can be anywhere;

Exercise:

- unconstrained problem, write a function $f(x)$ whose maximum is in $(x = 0)$
- unconstrained problem, write a function $f(x_1, x_2)$ whose maximum is in $(3,3)$
- Find the maximum of $f(x) = 6x - x^2$
- Draw $f(x) = 6x - x^2$

Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

The minimum/maximum can be anywhere;

Exercise:

- unconstrained problem, write a function $f(x)$ whose maximum is in $(x = 0)$
  - $f(x) = -x^2$
- unconstrained problem, write a function $f(x_1, x_2)$ whose minimum is in $(3,3)$
  - $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$
- Find the maximum of $f(x) = 6x - x^2$
  - There will be a maximum in $x = 3$ (why?)
- Draw $f(x) = 6x - x^2$

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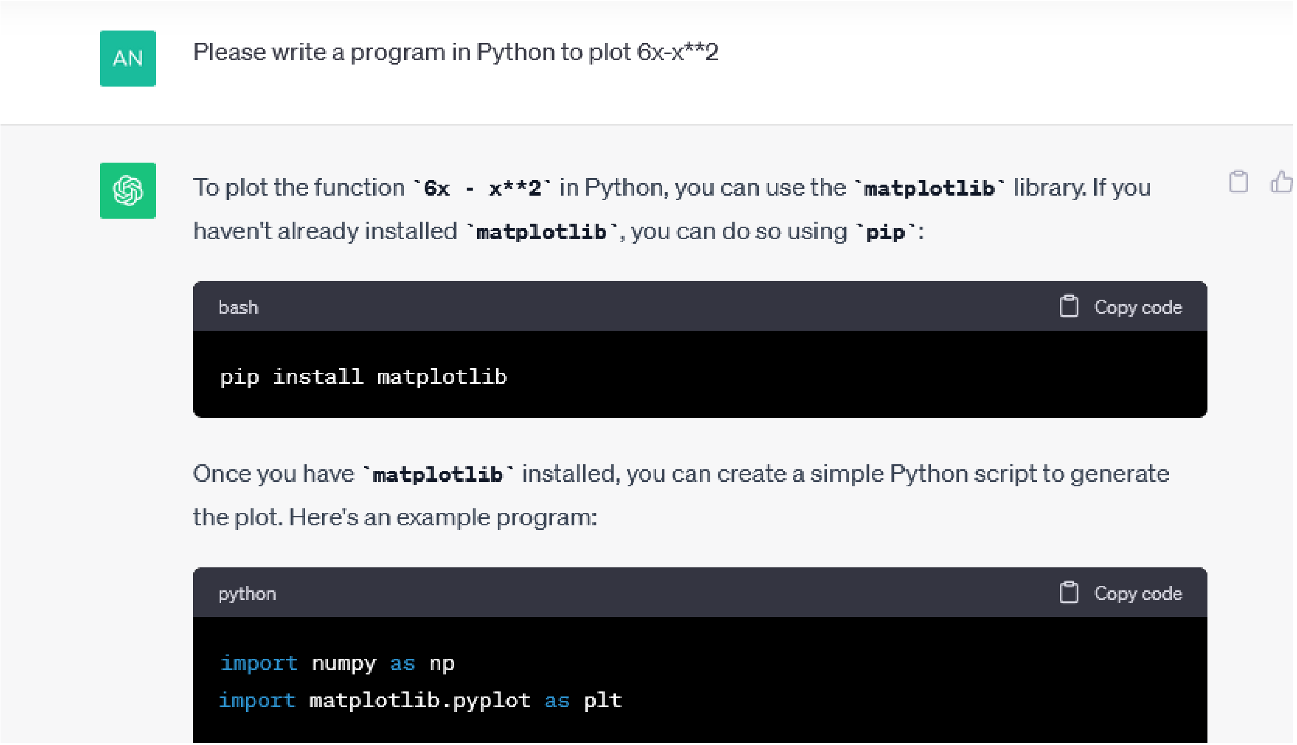

Please write a program in Python to plot 6x-x**2

To plot the function `6x - x**2` in Python, you can use the `matplotlib` library. If you haven't already installed `matplotlib`, you can do so using `pip`:

```bash
pip install matplotlib
```

Once you have `matplotlib` installed, you can create a simple Python script to generate the plot. Here's an example program:

```python
import numpy as np
import matplotlib.pyplot as plt
```

Ask ChatGTP

Go to

https://openai.com/

Login or create account

Ask ChatGPT to write a program in Python to plot $6x - x^2$

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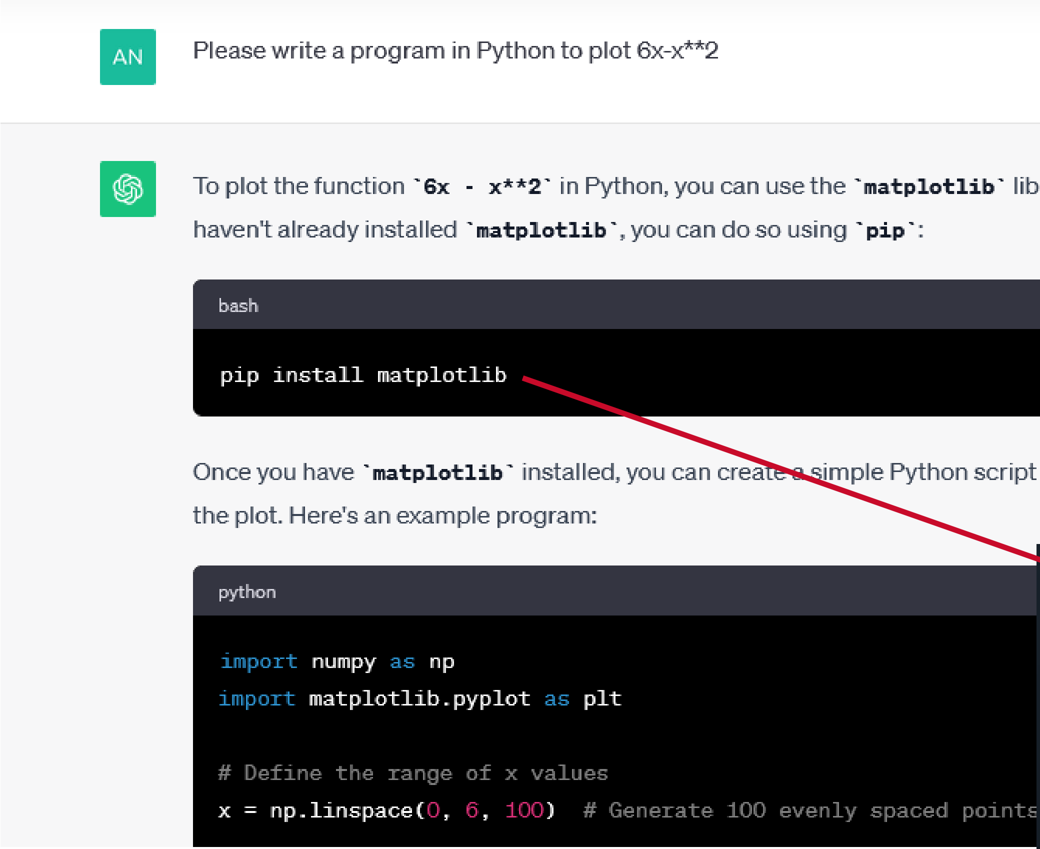


Please write a program in Python to plot 6x-x**2

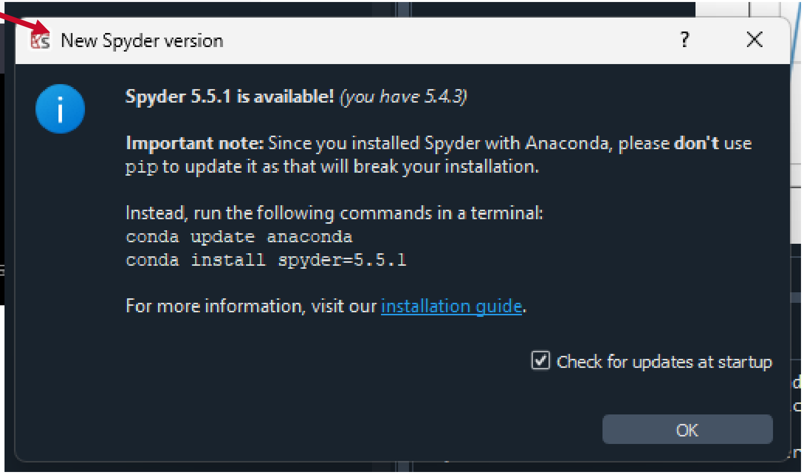
To plot the function `6x - x**2` in Python, you can use the `matplotlib` library. If you haven't already installed `matplotlib`, you can do so using `pip`:

```bash
pip install matplotlib
```

Once you have `matplotlib` installed, you can create a simple Python script to generate the plot. Here's an example program:

```python
import numpy as np
import matplotlib.pyplot as plt

# Define the range of x values
x = np.linspace(0, 6, 100)  # Generate 100 evenly spaced points
```

ChatGPT and Python disagree here – tell me what you find



Spyder 5.5.1 is available! *(you have 5.4.3)*

Important note: Since you installed Spyder with Anaconda, please **don't** use pip to update it as that will break your installation.

Instead, run the following commands in a terminal:

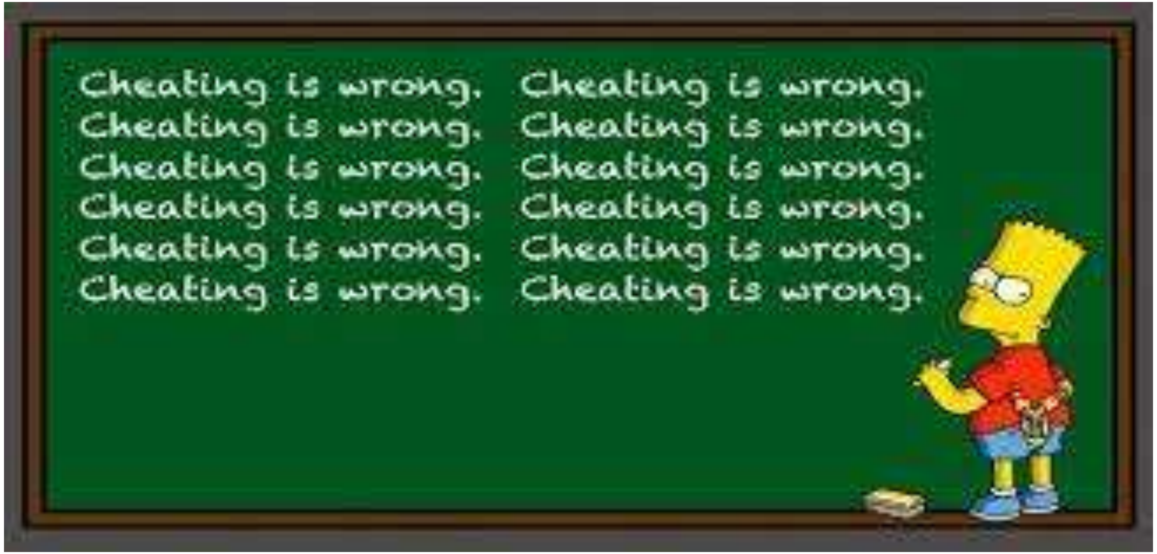
```
conda update anaconda
conda install spyder=5.5.1
```

For more information, visit our installation guide.

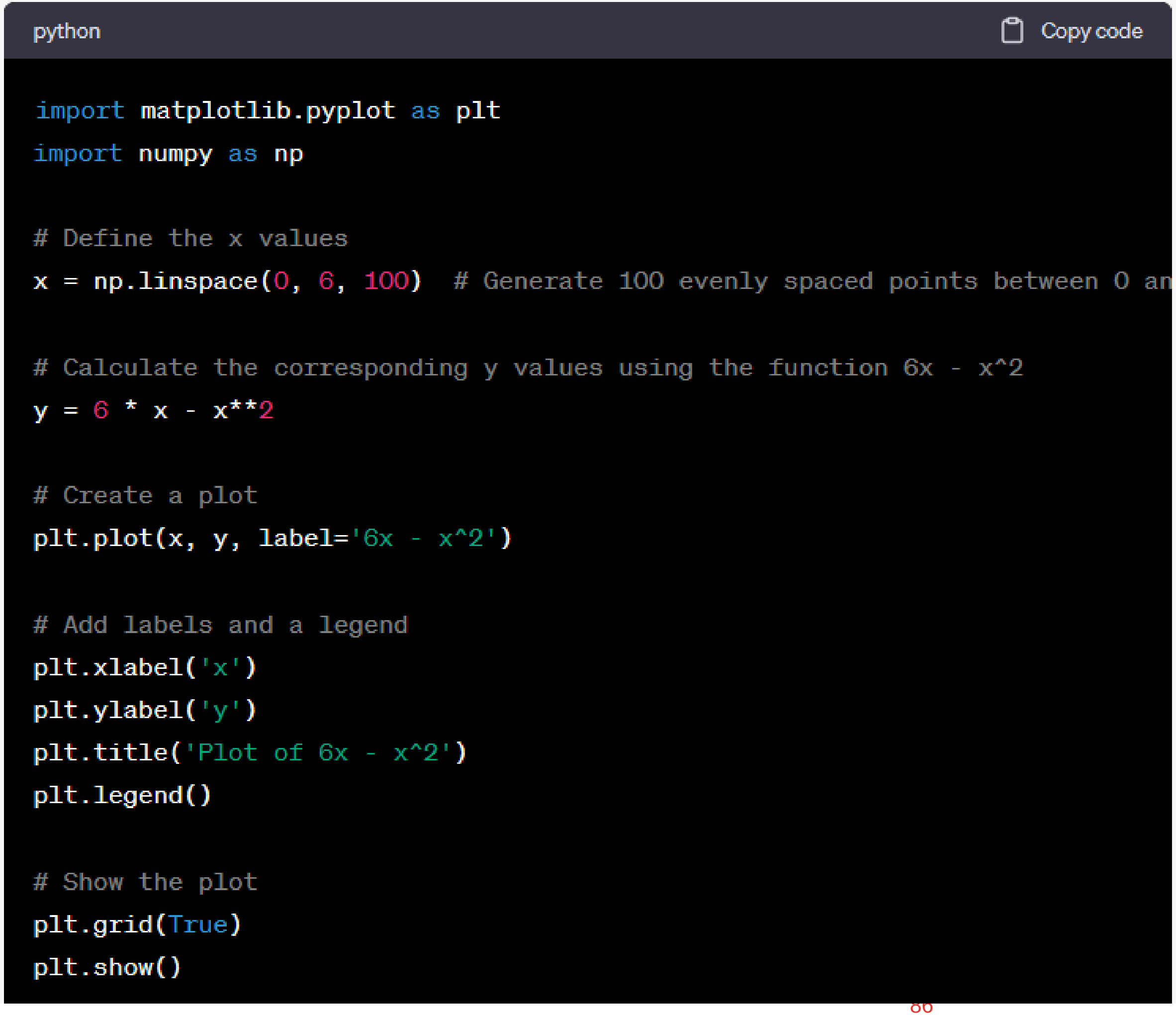
☑ Check for updates at startup

upf. BARCELONA SCHOOL OF MANAGEMENT

Is using ChatGPT cheating?


Cheating is wrong. Cheating is wrong.
Cheating is wrong. Cheating is wrong.
Cheating is wrong. Cheating is wrong.
Cheating is wrong. Cheating is wrong.
Cheating is wrong. Cheating is wrong.
Cheating is wrong. Cheating is wrong.

Source: The Simpsons, Twentieth Century Fox Film Corporation



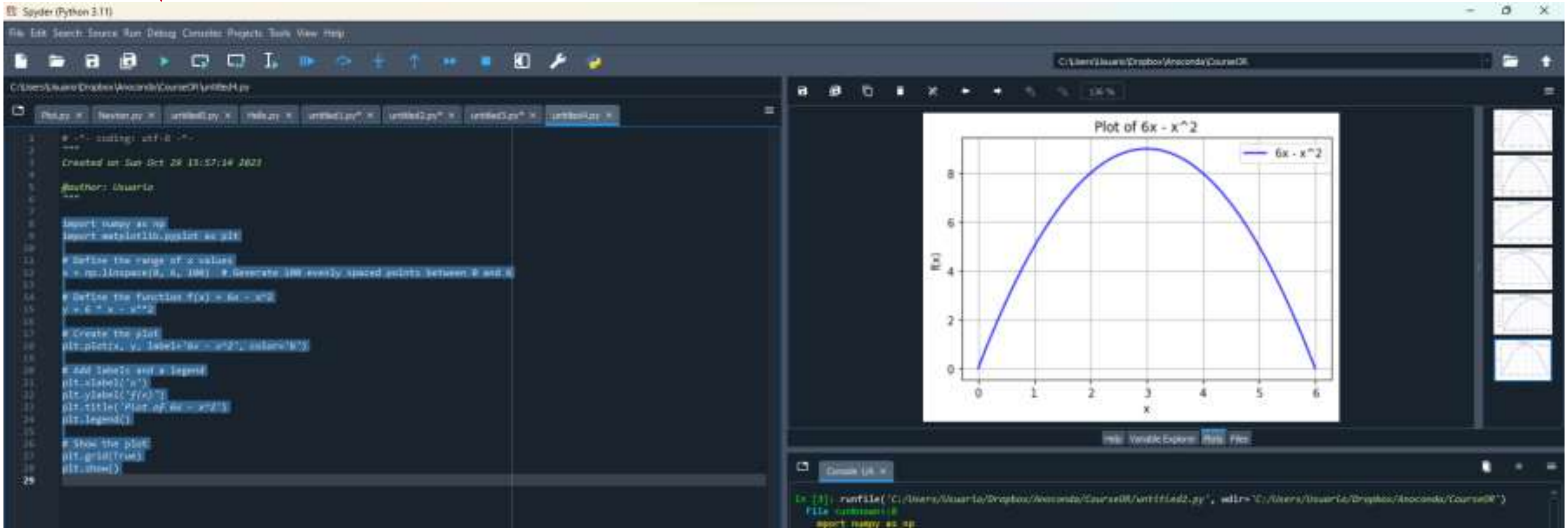
```python
import matplotlib.pyplot as plt
import numpy as np

# Define the x values
x = np.linspace(0, 6, 100)  # Generate 100 evenly spaced points between 0 an

# Calculate the corresponding y values using the function 6x - x^2
y = 6 * x - x**2

# Create a plot
plt.plot(x, y, label='6x - x^2')

# Add labels and a legend
plt.xlabel('x')
plt.ylabel('y')
plt.title('Plot of 6x - x^2')
plt.legend()

# Show the plot
plt.grid(True)
plt.show()
```

Install Anaconda from

https://www.anaconda.com/download

Launch Anaconda, launch Spyder, execute

Execute


Spyder (Python 3.11)
File Edit Search Source Run Debug Consoles Projects Tools View Help
Plot of 6x - x^2
6x - x^2
f(x)
x
0
2
4
6
8
0
1
2
3
4
5
6
Help Variable Explorer Plots Files

```
# -*- coding: utf-8 -*-
"""
Created on Sun Oct 29 13:57:14 2023

@author: Usuario
"""

import numpy as np
import matplotlib.pyplot as plt

# Define the range of x values
x = np.linspace(0, 6, 100)  # Generate 100 evenly spaced points between 0 and 6

# Define the function f(x) = 6x - x^2
y = 6 * x - x**2

# Create the plot
plt.plot(x, y, label='6x - x^2', color='b')

# Add labels and a legend
plt.xlabel('x')
plt.ylabel('f(x)')
plt.title('Plot of 6x - x^2')
plt.legend()

# Show the plot
plt.grid(True)
plt.show()
```

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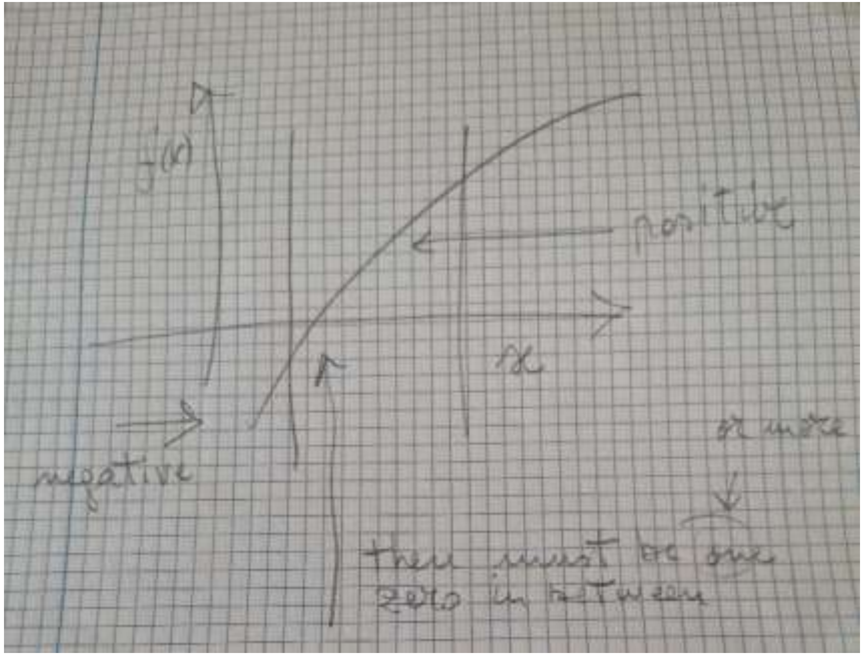
Bisection method (Bolzano search plan) – for concave functions.

If a continuous function has values of opposite sign inside an interval, then it has a root in that interval (Bolzano's theorem)

Not rocket science!



Bernardus Placidus
Johann Nepomuk
Bolzano, 1781–1848
Source: Wikipedia
Commons

Bisection method (Bolzano search plan) – for concave functions.

If a continuous function has values of opposite sign inside an interval, then it has a root in that interval (Bolzano's theorem)

Applying this theorem to the derivative $\frac{\partial f(x)}{\partial x}$, knowing that for concave functions the maximum corresponds to the point where $\frac{\partial f(x)}{\partial x} = 0$ one can first identify an interval where $\frac{\partial f(x)}{\partial x}$ changes sign, then reduce the dimension interval iteratively to get the solution

Bernardus Placidus
Johann Nepomuk
Bolzano, 1781–1848
Source: Wikipedia
Commons

Notation

x^* solution being sought

ε tolerance in the search of x^*

x' current trial solution

x_l current lower bound

x_u current upper bound

We want the maximum of

$$f(x) = 6x - x^2 - \frac{x^3}{3}$$

Setting the tolerance ε at 0.1 (10%)

Procedure: Find extreme x_u and x_l so that $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_l}$<0, while $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_u}$>0 and initialize $x' = \frac{x_l+x_u}{2}$

1) Evaluate $\frac{\partial f(x)}{\partial x}$ at $\mathrm{x} = x'$

2) if $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x\prime}$<0 redefine x_l= x'

3) if $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x\prime}$ >0 redefine x_u= x'

4) Update $x' = \frac{x_l+x_u}{2}$

Iterate 1–4 till $x_u - x_l \leq 2\varepsilon$ so that new x'must be within ε of x^*

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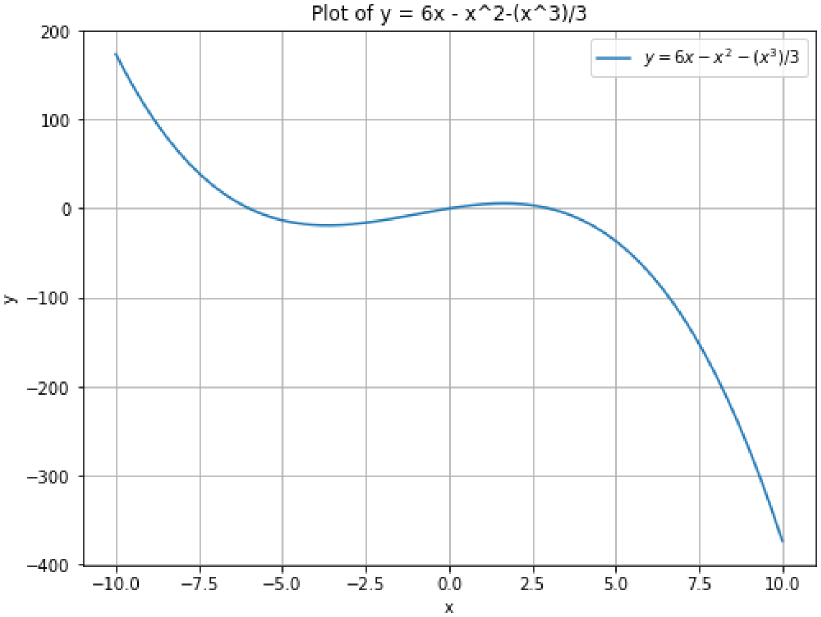
Use the procedure to find a local maximum of

$$f(x) = 6x - x^2 - \frac{x^3}{3}$$

Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

Between 1. and 2. Set the tolerance ε at 0.1 (10%)

Help: $\frac{\partial f(x)}{\partial x} = 6 - 2x - x^2$



Use the procedure to find in interval (1.,2.) the maximum of

$$f(x) = 6x - x^2 - \frac{x^3}{3}$$

$$\frac{\partial f(x)}{\partial x} = 6 - 2x - x^2$$

Set the tolerance ε at 0.1 (10%)

By hand, I get x=1.72, $\frac{\partial f(x)}{\partial x}$ ~0.4 after 3 iterations and some number crunching; search interval $x \in (1,2)$

The procedure is laborious (slow convergence)

Procedure

Find extreme x_u and x_l so that $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_l}$<0, while $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_u}$>0 and initialize $x' = \frac{x_l+x_u}{2}$

1) Evaluate $\frac{\partial f(x)}{\partial x}$ at $x = x'$

2) if $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x'}$<0 redefine $x_l = x'$

3) if $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x'}$ >0 redefine $x_u = x'$

4) Update $x' = \frac{x_l+x_u}{2}$

Iterate 1−4 till $x_u - x_l \leq 2\varepsilon$ so that new x'must be within ε of x^*

Newton's method – for concave functions

Newton is credited to have discovered calculus (in parallel with Leibniz) and his method is that of a quadratic approximation based on a truncated Taylor series

Isac Newton (1643-1727)

Source: Wikipedia Commons

$$f(x_{i+1}) = f(x_i) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}(x_{i+1}-x_i) + \frac{1}{2}\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}\big((x_{i+1}-x_i)\big)^2 +$$

$$\frac{1}{6}\left.\frac{\partial^3 f(x)}{\partial x^3}\right|_{x=x_i}\big((x_{i+1}-x_i)\big)^3 \cdots + \frac{1}{n!}\left.\frac{\partial^n f(x)}{\partial x^n}\right|_{x=x_i}\big((x_{i+1}-x_i)\big)^n$$

We stop (truncate)
now at the second
order term

Looking at this as a function of only x_{i+1}, with x_i and its derivative as fixed

$$f(x_{i+1}) = f(x_i) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i} (x_{i+1} - x_i) + \frac{1}{2} \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i} (x_{i+1} - x_i)^2 + \cdots$$

We can differentiate with respect to x_{i+1} to get

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_{i+1}} = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i} + \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i} (x_{i+1} - x_i)$$

Setting this to zero (as to find the maximum) and reordering we get

$$x_{i+1} = x_i - \frac{\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}}{\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}}$$

Source: The Simpson, 20th Television Animation (The Walt Disney Company)

Using this beauty $x_{i+1} = x_i - \dfrac{\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}}{\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}}$

The procedure is simple: find a trial value x_i by inspection

1) Evaluate $\left.\dfrac{\partial f(x)}{\partial x}\right|_{x=x_i}$

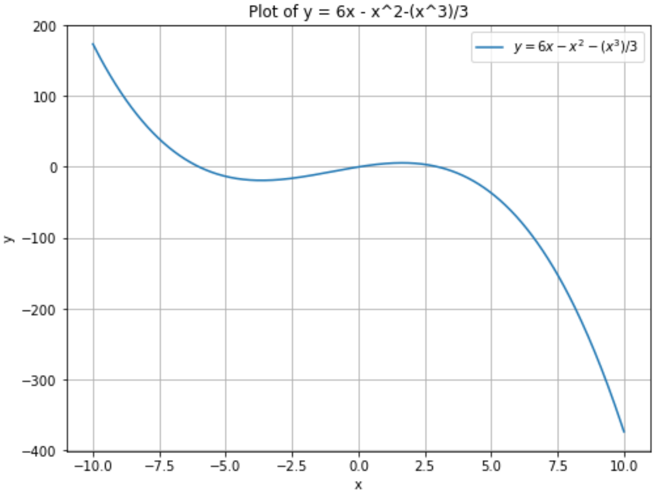
2) Evaluate $\left.\dfrac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}$

3) Compute x_{i+1} as from the formula above

Iterate 1–3 till $|x_{i+1} - x_i| \leq \varepsilon$

Exercise: Compute at least up to the first iteration, starting from $x_i = 1$ (at home, facultative)

$$f(x) = 6x - x^2 - \frac{x^3}{3}; \qquad \frac{\partial f(x)}{\partial x} = 6 - 2x - x^2; \qquad \frac{\partial^2 f(x)}{\partial x^2} = -2 - 2x$$



Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

Using this beauty $x_{i+1} = x_i - \frac{\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}}{\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}}$

The procedure is simple: find a trial value x_i by inspection

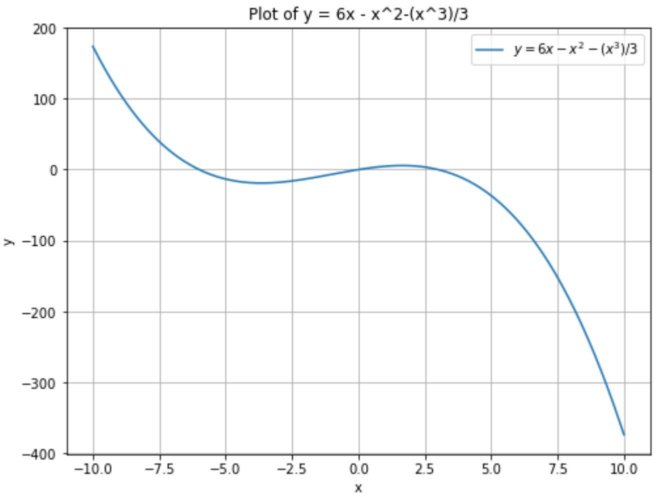
1) Evaluate $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}$

2) Evaluate $\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}$

3) Compute x_{i+1} as from the formula above

Iterate 1–3 till $|x_{i+1} - x_i| \leq \varepsilon$, starting from $x_i = 1$

Phyton code can be used again

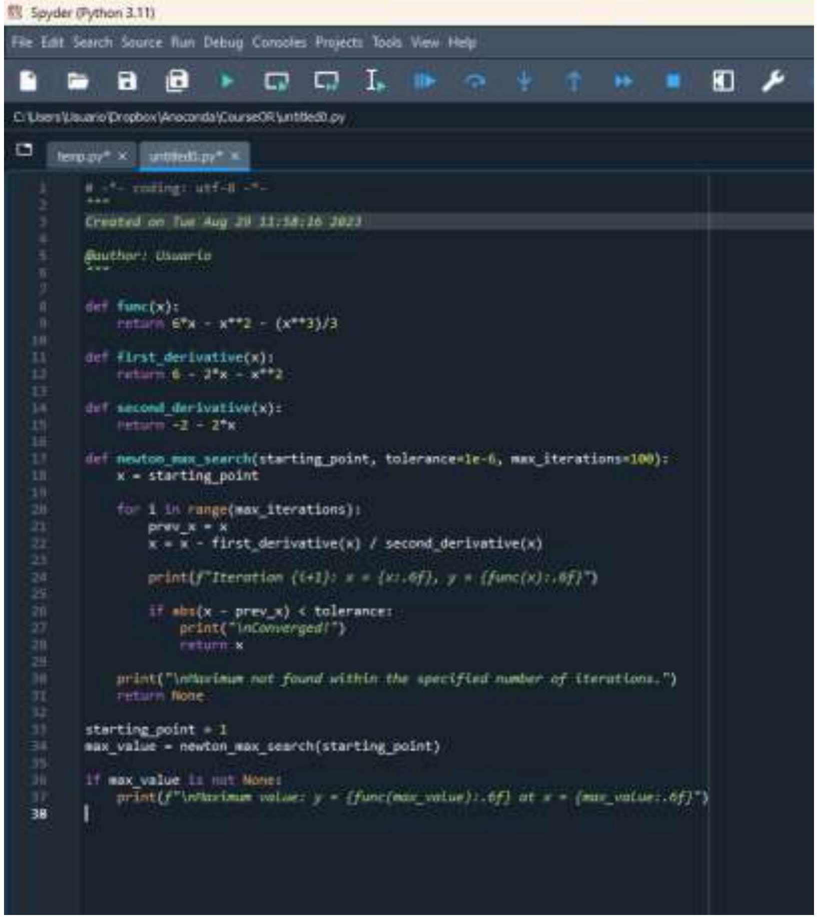


Question: Please find the maximum of the function y=6x−x**2-(x**3)/3, using the method of Newton pure, using the first and the second derivative, with starting point x=1, and printing out all steps of the search.

To find the maximum of the function $y = 6x - x^2 - \frac{x^3}{3}$ using the Newton-Raphson method with the first and second derivatives, you can follow these steps:

1. Define the function $y = 6x - x^2 - \frac{x^3}{3}$.
2. Calculate the first and second derivatives of the function.
3. Implement the Newton-Raphson method using the formula: $x_{\text{new}} = x_{\text{old}} - \frac{f'(x_{\text{old}})}{f''(x_{\text{old}})}$.
4. Iterate until the change in x is smaller than a certain tolerance or a maximum number of iterations is reached.
5. Print out each iteration's x value and the corresponding function value.

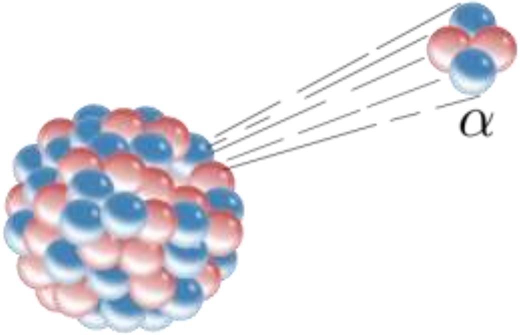
Here's how you can implement this in Python:


Spyder (Python 3.11)
File Edit Search Source Run Debug Consoles Projects Tools View Help
C:\Users\Usuario\Dropbox\Anaconda\CourseOR\untitled0.py
temp.py*
untitled0.py*

```python
# -*- coding: utf-8 -*-
"""
Created on Tue Aug 29 11:58:16 2023

@author: Usuario
"""

def func(x):
    return 6*x - x**2 - (x**3)/3

def first_derivative(x):
    return 6 - 2*x - x**2

def second_derivative(x):
    return -2 - 2*x

def newton_max_search(starting_point, tolerance=1e-6, max_iterations=100):
    x = starting_point

    for i in range(max_iterations):
        prev_x = x
        x = x - first_derivative(x) / second_derivative(x)

        print(f"Iteration {i+1}: x = {x:.6f}, y = {func(x):.6f}")

        if abs(x - prev_x) < tolerance:
            print("\nConverged!")
            return x

    print("\nMaximum not found within the specified number of iterations.")
    return None

starting_point = 1
max_value = newton_max_search(starting_point)

if max_value is not None:
    print(f"\nMaximum value: y = {func(max_value):.6f} at x = {max_value:.6f}")
```

```
Iteration 1: x = 1.750000,  y = 5.651042
Iteration 2: x = 1.647727,  y = 5.680162
Iteration 3: x = 1.645752,  y = 5.680173
Iteration 4: x = 1.645751,  y = 5.680173
```

A different model, where we explore rather than optimize



Source: Wikipedia common

The Bateman equations, initially written for a radioactive decay chain, can be used to describe any process where transfer occur among compartments without backward term

$\partial N_1 / \partial t = -\lambda_1 N_1$ N_1 concentration of species 1, disappear proportionally to itself …

$\partial N_2 / \partial t = \lambda_1 N_1 - \lambda_2 N_2$ …to become N_2 that in turn disappear proportionally to itself…

… …and so on…

$\partial N_n / \partial t = \lambda_{n-1} N_{n-1} - \lambda_n N_n$ …till and end−product is reached

We choose this as it is nonlinear and the differential equation has a neat analytic solution

$\partial N_1 / \partial t = -\lambda_1 N_1$

$\partial N_2 / \partial t = \lambda_1 N_1 - \lambda_2 N_2$

…

$\partial N_n / \partial t = \lambda_{n-1} N_{n-1} - \lambda_n N_n$

For $N_1(0) \neq 0$, $N_i(0) = 0 \ \forall i \neq j$ the solution is

$$N_k(t) = \frac{N_1(0)}{\lambda_k} \sum_{i=1}^{k} \lambda_i \alpha_i \, e^{-\lambda_i t}$$

With

$$\alpha_i \prod_{j=1, j \neq i}^{k} \frac{\lambda_i}{\lambda_j - \lambda_i}$$

Interested in playing with this function in Python? Script in eCampus

A last model, where we revisit our simple linear form (from Lesson 3): $y = \sum_{i=1}^{k} \Omega_i Z_i$

Where y (a scalar) is the output of interest, the Ω_i's were fixed coefficients and Z_i's are uncertain input factors following a Normal distribution

$Z_i \sim N\left(\bar{z}_i, \sigma_{Z_i}\right)$

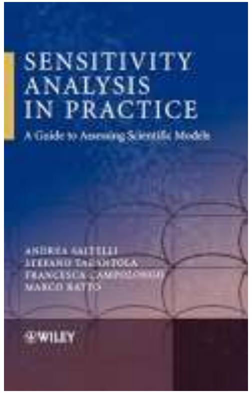
Where $\bar{z}_i = 0, i = 1,2, \dots k$ are the means of the factors Z_i's and σ_i their standard deviations

We now allow the Ω_i to be uncertain as well

$\Omega_i \sim N\left(\overline{\Omega_i}, \sigma_{\Omega_i}\right)$ where $\overline{\Omega_i} = 0, i = 1,2, \dots k$ are the means of the factors Ω_i's and σ_{Ω_i} their standard deviations

Interested in playing with this function in Python? Script in eCampus

In this book we took $\bar{z}_i$ to be zero, and called $y = \sum_{i=1}^{k} \Omega_i Z_i$ a balanced portfolio, where the Z_i are the assets and the Ω_i the amount held of each security



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There are as well stochastic search method (called meta-heuristics in the Hillier's book) where the search is done iteratively with trial points and rules to point the search in the right direction, without being greedy

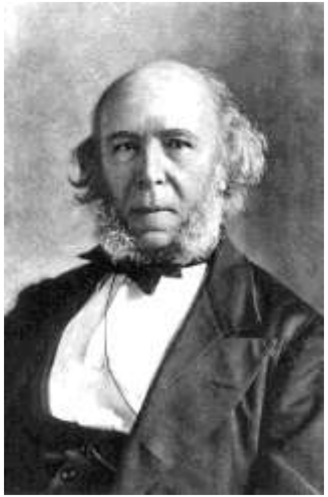
- Tabu Search (don't go there if you have been there already)
- Simulated Annealing (you can walk in the wrong direction but with lower probability)
- Genetic Algorithms (let the fitter reproduce themselves)

We study this

Genetic algorithms (let the **fitter** reproduce themselves)

Note: Darwin used ‘fittest’, borrowing the term from Spencer, but ‘fitter’ is more apt to his theory as well as to what genetic algorithms do.

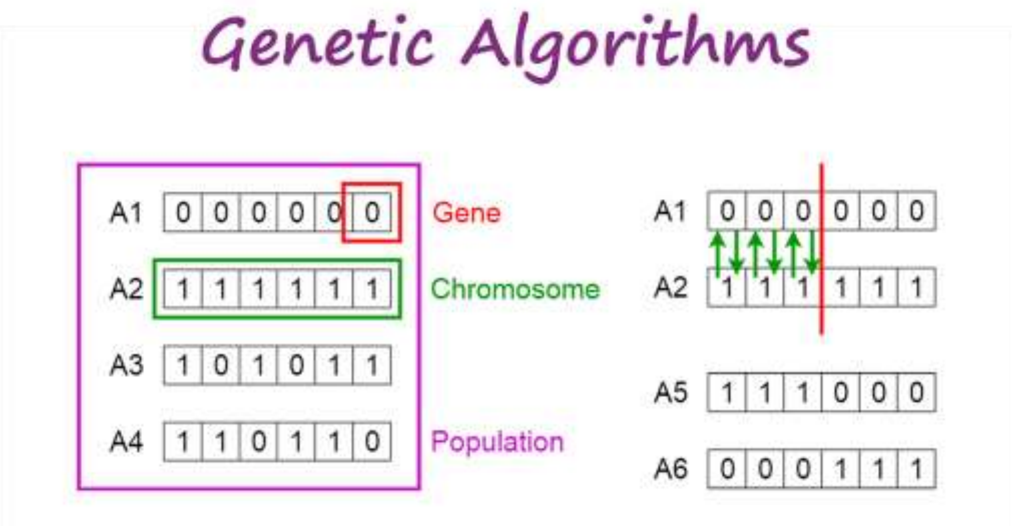
“Darwin did not consider the process of evolution as the survival of the fittest; he regarded it as survival of the fitter, because the “struggle for existence” is relative and thus not absolute. Instead, the winners with respect to species within ecosystems could become losers with a change of circumstances” (https://www.britannica.com)



Herbert Spencer
(1820–1903)

Charles Darwin
(1809–1882)

Unlike Tabu search and Simulated Annealing, genetic algorithms do not work with a wandering point, but with an evolving population – a collection of candidate points is generated right at the start, then these generate offspring

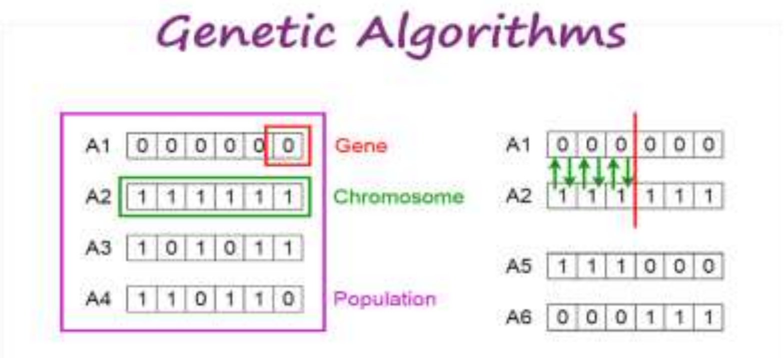


Source: https://towardsdatascience.com/

Parents

Children

At each generation the parents with the higher fitness have higher probability of reproducing, with each parent passing part of his genes to the offspring ➔ fitness in terms of objective function Z

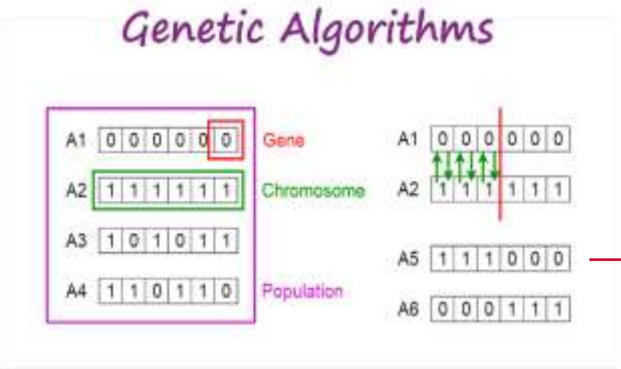


Parents

Children

Source: https://towardsdatascience.com/

Additional random mutations can occur ('errors' in the transcription of DNA or epigenetic factors in the genetic metaphor)



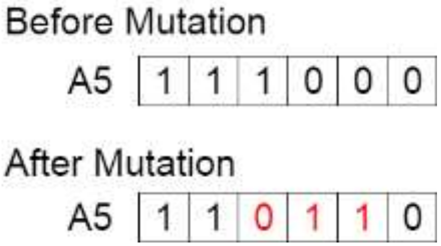
A1 0 0 0 0 0 0
A2 1 1 1 1 1 1
A3 1 0 1 0 1 1
A4 1 1 0 1 1 0

A1 0 0 0 0 0 0
A2 1 1 1 1 1 1

A5 1 1 1 0 0 0
A6 0 0 0 1 1 1

Parents

Children


Before Mutation

A5 1 1 1 0 0 0

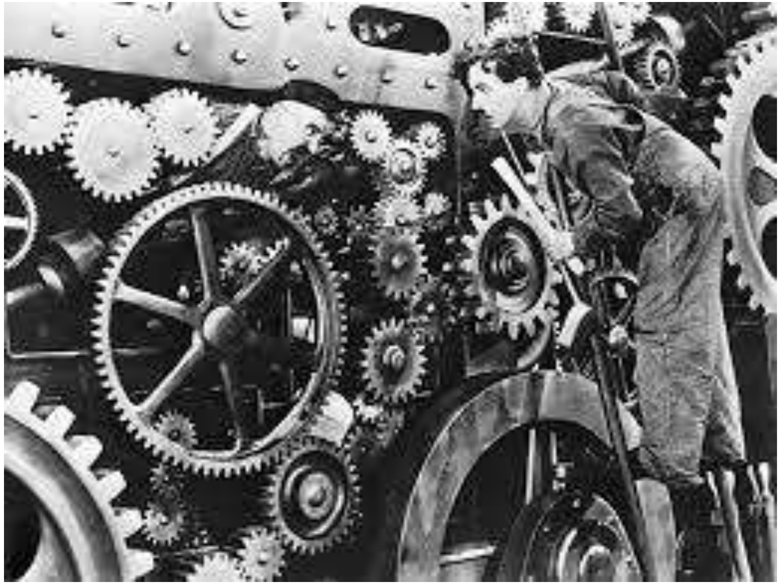
After Mutation

A5 1 1 0 1 1 0

Source: https://towardsdatascience.com/

Genetic algorithms have lots of movable parts! In order to use them you must decide

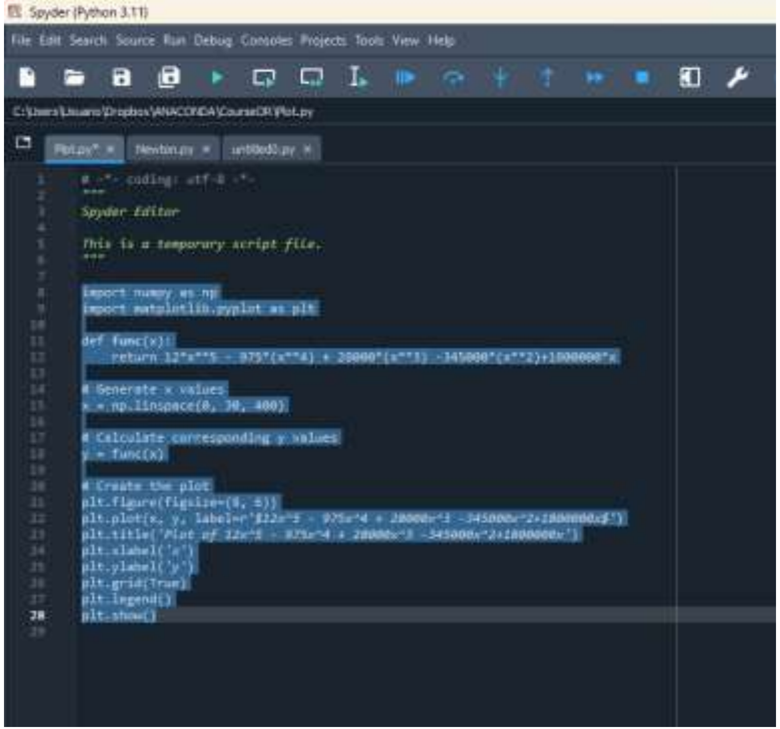
- Size and composition of the initial population
- How to select the parents based on Z
- How to exchange the genes of the parent to generate the children
- Mutation rate
- Stopping rule



Source: Charlie Chaplin's Modern Times

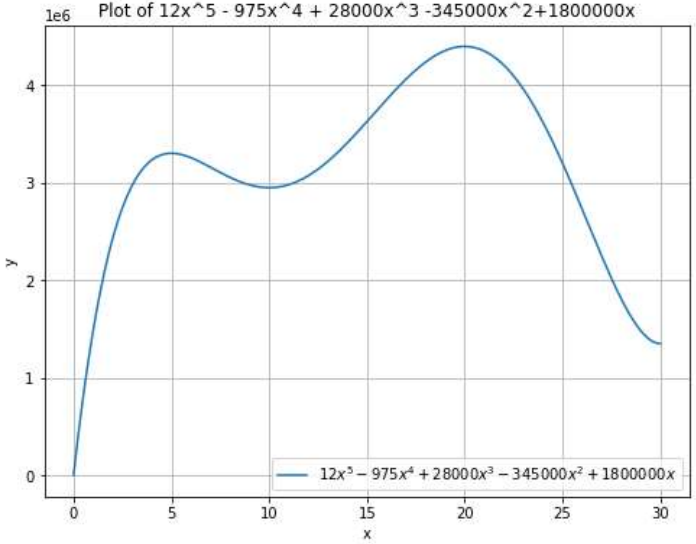
Try with GA with the monster (Hillier, Chapter #14)

$$y = 12x^5 - 975x^4 + 28{,}000x^3 - 345{,}000x^2 + 1{,}800{,}000x^5$$



Spyder (Python 3.11)
File Edit Search Source Run Debug Consoles Projects Tools View Help
C:\Users\Usuario\Dropbox\ANACONDA\CourseOR\Plot.py
Plot.py*
Newton.py
untitled0.py

```
# -*- coding: utf-8 -*-
"""
Spyder Editor

This is a temporary script file.
"""

import numpy as np
import matplotlib.pyplot as plt

def func(x):
    return 12*x**5 - 975*(x**4) + 28000*(x**3) -345000*(x**2)+1800000*x

# Generate x values
x = np.linspace(0, 30, 400)

# Calculate corresponding y values
y = func(x)

# Create the plot
plt.figure(figsize=(8, 6))
plt.plot(x, y, label=r'$12x^5 - 975x^4 + 28000x^3 -345000x^2+1800000x$')
plt.title('Plot of 12x^5 - 975x^4 + 28000x^3 -345000x^2+1800000x')
plt.xlabel('x')
plt.ylabel('y')
plt.grid(True)
plt.legend()
plt.show()
```



We trick the monster by adding the constraint the x must be integer

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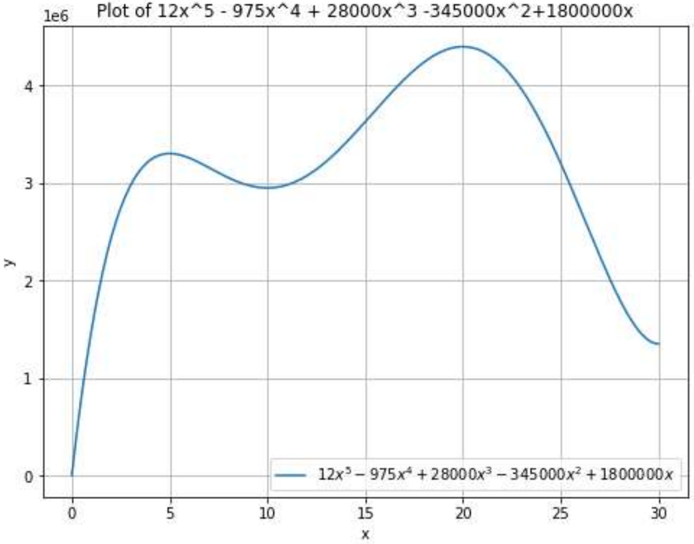
Try with GA with the monster (Hillier, Chapter #14)

$$y = 12x^5 - 975x^4 + 28{,}000x^3 - 345{,}000x^2 + 1{,}800{,}000x^5$$

Since the x axis spans from zero to 32, we can represent the possible solutions in a nice binary notation



00=000000
01=000001=2^0
02=000010=2^1
03=000011=$2^1 + 2^0$
04=000100=2^2
…
07=000111=$2^2 + 2^1 + 2^0$
08=001000=2^3
…
15=001111=$2^3 + 2^2 + 2^1 + 2^0$
16=010000=2^4
…
31=011111=$2^4 + 2^3 + 2^2 + 2^1 + 2^0$
32=100000=2^5

Source: Wikipedia Commons

Leonardo of Pisa,
said Fibonacci
1170-1250

We trick the monster by adding the constraint the x must be integer – **but we refresh binary numbers first**

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Never met binary?

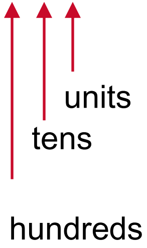
$00=000000$
$01=000001=2^0$
$02=000010=2^1$
$03=000011=2^1+2^0$
$04=000100=2^2$
…
$07=000111=2^2+2^1+2^0$
$08=001000=2^3$
…
$15=001111=2^3+2^2+2^1+2^0$
$16=010000=2^4$
…
$31=011111=2^4+2^3+2^2+2^1+2^0$
$32=100000=2^5$
$64\ =1000000=2^6$
$128=10000000=2^7$

Take a way to write a number

CCXXVI

This is conceptually **far** from

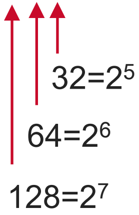
226



units
tens
hundreds

While this is conceptually close to

11100010



$32=2^5$
$64=2^6$
$128=2^7$

Never met binary?

0.1=1/2
0.01=1/4
0.001=1/8

0.111=0.875

11.001=3.125

…

Take a way to write a number

CCXXVI

This is conceptually **far** from

226

units
tens
hundreds

While Arabic and binary are conceptually close

11100010

$32=2^5$
$64=2^6$
$128=2^7$

Then we have to cook some rules, e.g.

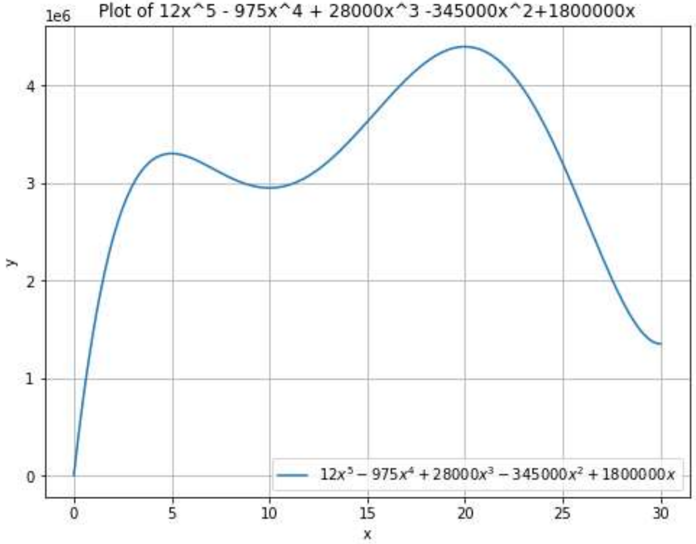
Starting population: 10 individuals (for this problem)

Choose the five fittest and the two lest fit for **matching** (coupling randomly)

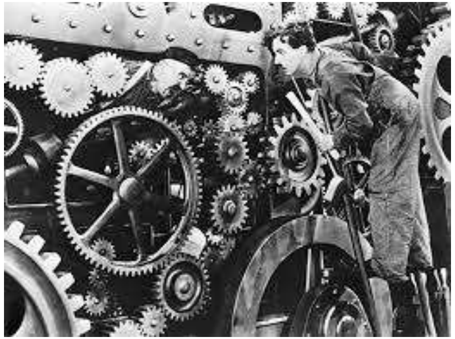
Switching the genes: keep repeated genes and switch the different ones

Mutation rate 1/10

Stopping rule after five iterations without improvements



Lots of moving parts; these are the choices suggested in Hillier but other choices are possible



Source: Charlie Chaplin’s Modern Times

One possible way of matching

Parents
011000
110101

Repeated genes are passed to the next generation

Children
x1xx0x
x1xx0x

How to choose the missing x's? **Russian roulette**

x1xx0x
x1xx0x

Generate a random number between 0 and 1; if between 0 and .4999 replace the first x with a zero, if between .5 and 1 replace it with a one

Source: From movie The Deer Hunter, Source: Wikipedia commons

Source: https://www.gettyimages.es

Source: https://wordwall.net/

Where the name comes from – from revolver to spinning wheel9

How to choose the x's? **Russian roulette**

x1xx0x
x1xx0x

Generate a random number between 0 and 1; if between 0 and .4999 replace the first x with a zero, if between .5 and 1 replace it with a one

This was done and we have children

011000
010101

How to mutate 10% of this? Russian roulette: if the random number is between 0 and 0.0999 choose to mutate, otherwise no mutations and pass to the next couple;

Russian roulette also to decide which gene to change; if there are 12 (6+ 6) candidate genes (two parent of five genes each), **how do we do this time?**

If between 0 and $\frac{1}{12}$ mutate the first gene of the first child, in between $\frac{1}{12}$ and $\frac{2}{12}$ the second gene of the fist child ⋯ if between $\frac{11}{12}$ and 1 the last gene of the second child

Source: https://www.gettyimages.es

Source: https://wordwall.net/

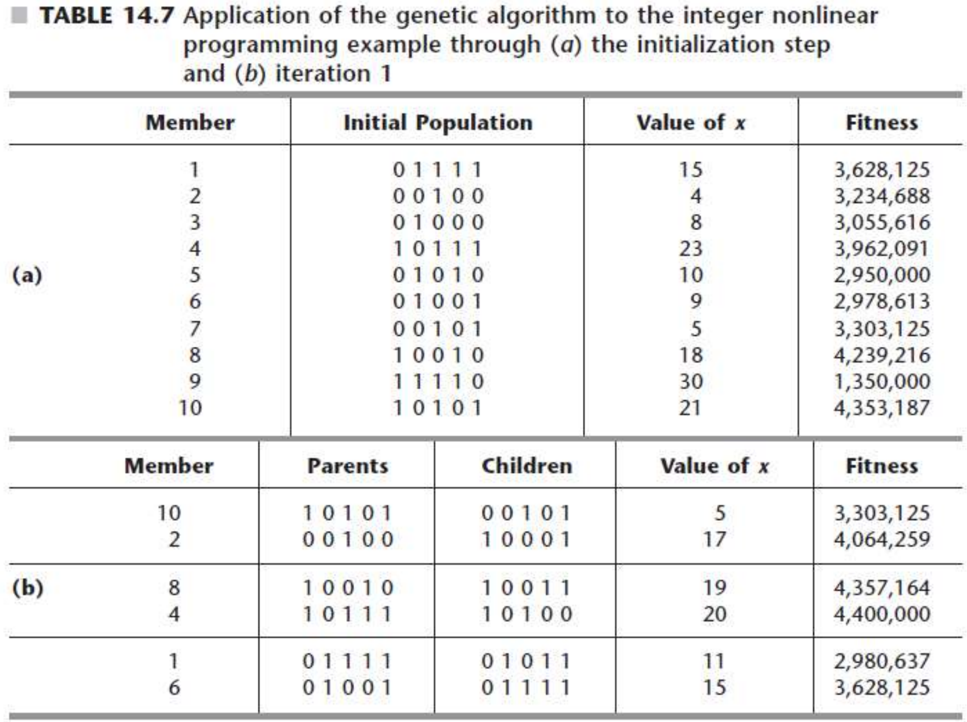
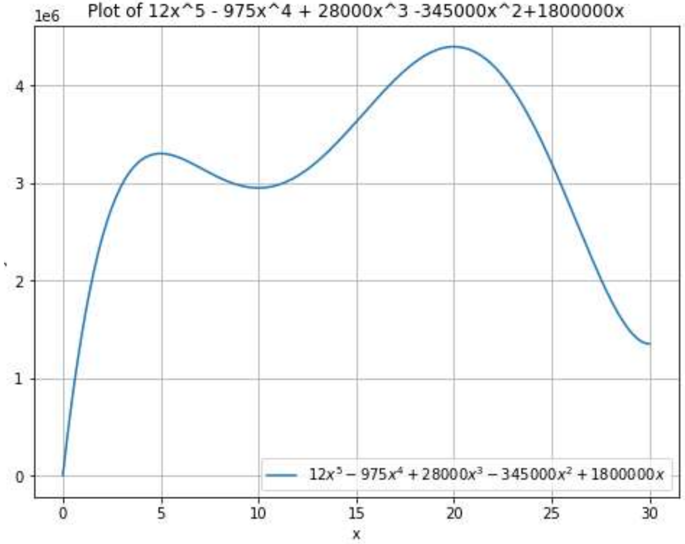


TABLE 14.7 Application of the genetic algorithm to the integer nonlinear programming example through (*a*) the initialization step and (*b*) iteration 1

	Member	Initial Population	Value of x	Fitness
(a)	1	0 1 1 1 1	15	3,628,125
	2	0 0 1 0 0	4	3,234,688
	3	0 1 0 0 0	8	3,055,616
	4	1 0 1 1 1	23	3,962,091
	5	0 1 0 1 0	10	2,950,000
	6	0 1 0 0 1	9	2,978,613
	7	0 0 1 0 1	5	3,303,125
	8	1 0 0 1 0	18	4,239,216
	9	1 1 1 1 0	30	1,350,000
	10	1 0 1 0 1	21	4,353,187

	Member	Parents	Children	Value of x	Fitness
(b)	10	1 0 1 0 1	0 0 1 0 1	5	3,303,125
	2	0 0 1 0 0	1 0 0 0 1	17	4,064,259
	8	1 0 0 1 0	1 0 0 1 1	19	4,357,164
	4	1 0 1 1 1	1 0 1 0 0	20	4,400,000
	1	0 1 1 1 1	0 1 0 1 1	11	2,980,637
	6	0 1 0 0 1	0 1 1 1 1	15	3,628,125



All this is very nice to code; here the starting and first iteration from the Hillier book

And the convergence if rapid for this simple case, good results already at the fist iteration

The procedure can of course be applied to non-integer numbers, as these can as well be written in binary notation

Exercise: write 412 in binary

412=110011100

256y 128y 64n 32n 16y 8y 4y 2n 1n

I start by 256 because the next power of 2 (512) is too big; **get 1**

412-256=156; so I can fit in 128; **get 11**

The difference is 28, so I cannot fit a 64: **get 110**

Cannot fit a 32, **get 1100**

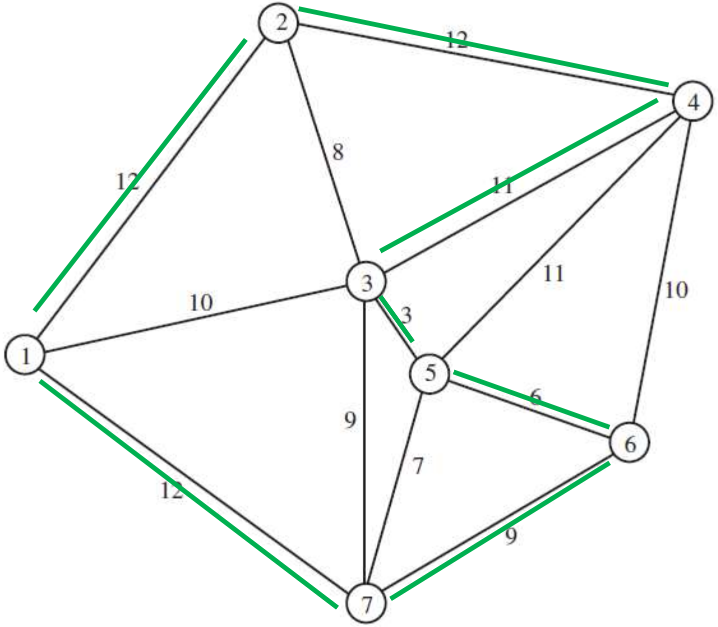
Can fit 16,8,4 add to 28 **get 1100111**

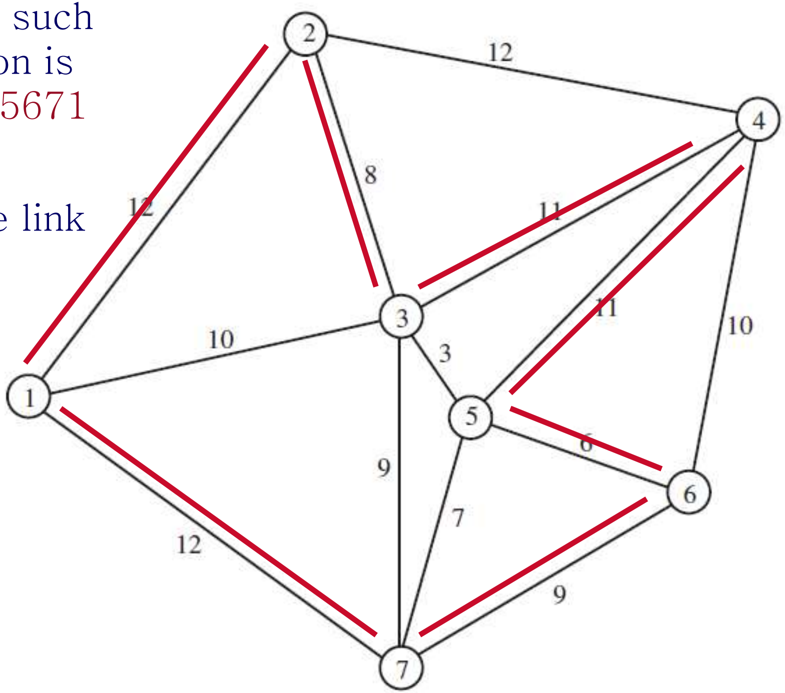
No need of 2 and 1 the last two powers, **get 110011100**

Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

Genetic algorithms can also be applied to problems such as the traveling salesman; in this case the population is constituted by candidate trajectories, such as 12345671 and 12435671

A child of this couple can inherit the link 2−3 or the link 2−4 ···





Homework

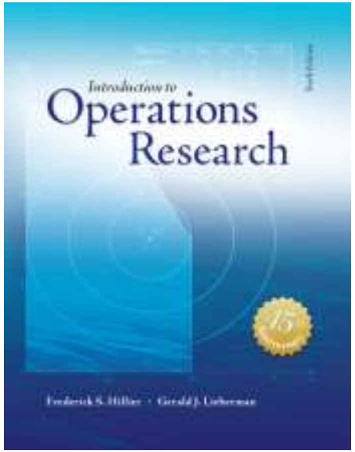
1) Install Anaconda on your laptop and bring it to class https://www.anaconda.com/download

2) Read pages 208-225 of the Mann book and solve all exercises from 5.41 to 5.44. It is not forbidden to use Excel.

3) Solve Hillier online book problem 12.1.3 page 534, only question (a) Formulate a BIP model for this problem.

4) Solve Hillier online book problem 12.1-4. page 534, only question (a) Formulate a BIP model for this problem.

5) Solve Hillier online book problem 12.3-1. page 535, only question (a) Formulate a BIP model for this problem.



Mann, P.S. (2010) Introductory Statistics. 7th edition. Wiley. https://www.andreasaltelli.eu/file/repository/Mann_Introductory_Statistics_7th_Ed_.pdf

Hillier and Lieberman (2015) Operation Research (10th edition) https://www.andreasaltelli.eu/file/repository/Introduction_to_Operations_Research_10th_Frederick_S_Hillier.pdf

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Thank you

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