Máster Universitario en Administración y Dirección de Empresas
Full Time MBA

Quantitative methods for decision making

Professor Andrea Saltelli

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Elements of quantification for decision making with emphasis on operation research

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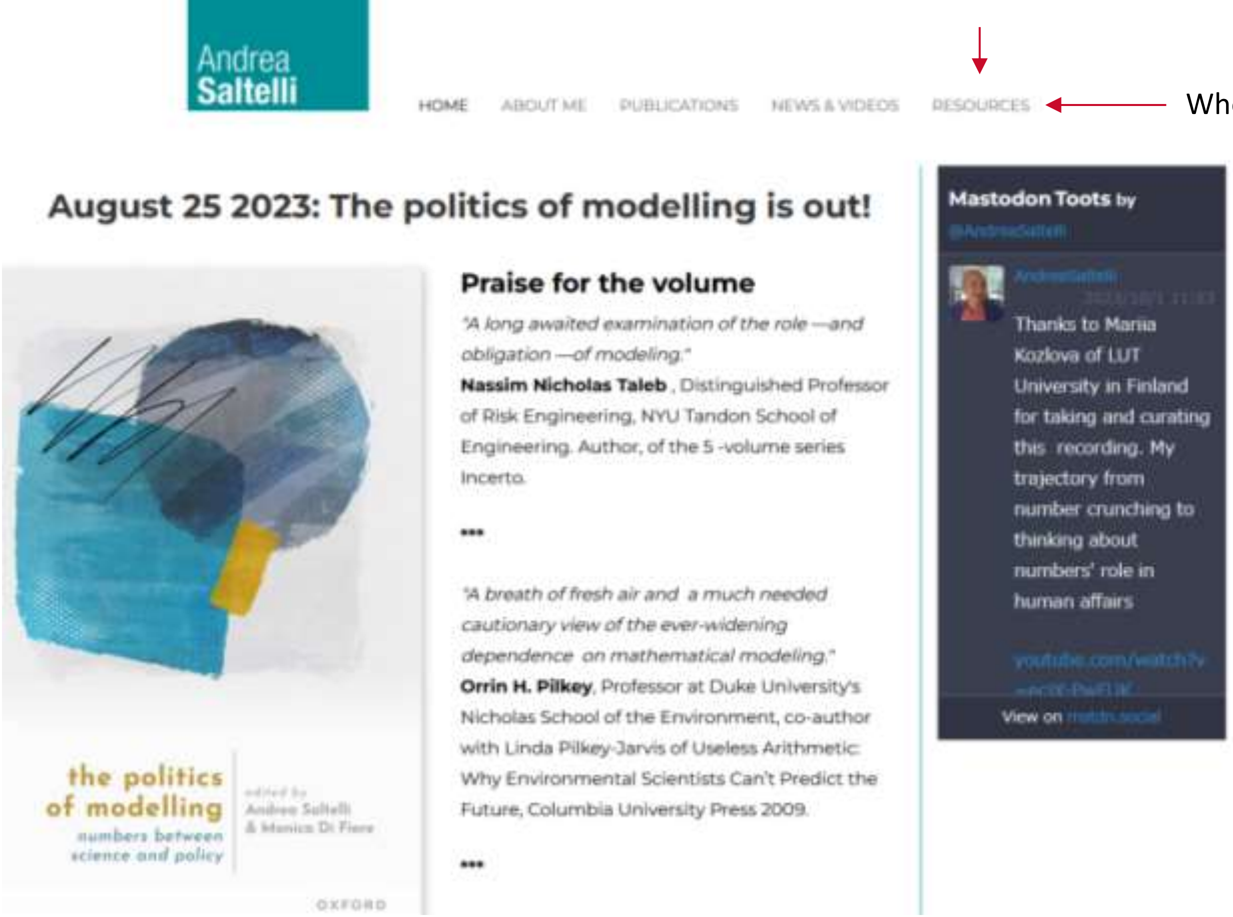

Where to find this talk

The talk is also at

https://ecampus.bsm.upf.edu/,

where you find additional reading material

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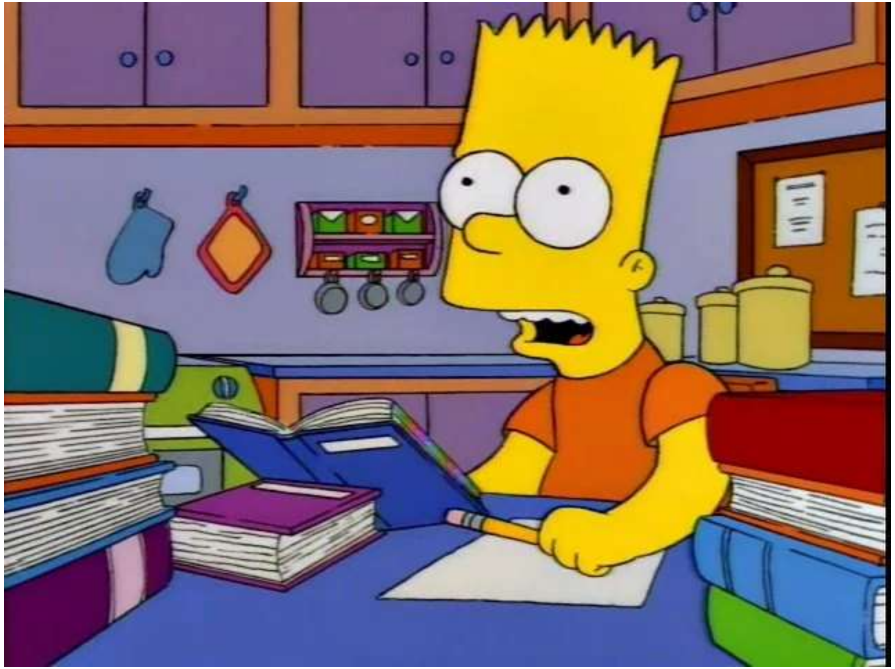
Mid term assignment October 29

9.00-10.40 lesson
10.40-11.00 break
11.00-13.00 assignment



Homework from last lesson

1. Compute the chance of having exactly 5 heads throwing a coin 8 times.

$$P(n,k,p) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{8}{5}\left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{8-5} = .219$$

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Homework

2. Re-do manually the exercise of slides 25-27 (Lesson #3) with these new values for the constraints

OLD	→	NEW
Maximize $Z = 3x_1 + 5x_2$		Maximize $Z = 3x_1 + 5x_2$
Subject to:		Subject to:
$x_1 \leq 4$		$x_1 \leq 5$
$2x_2 \leq 12$		$2x_2 \leq 13$
$3x_1 + 2x_2 \leq 18$		$3x_1 + 2x_2 \leq 19$
$x_1 \geq 0$		$x_1 \geq 0$
$x_2 \geq 0$		$x_2 \geq 0$

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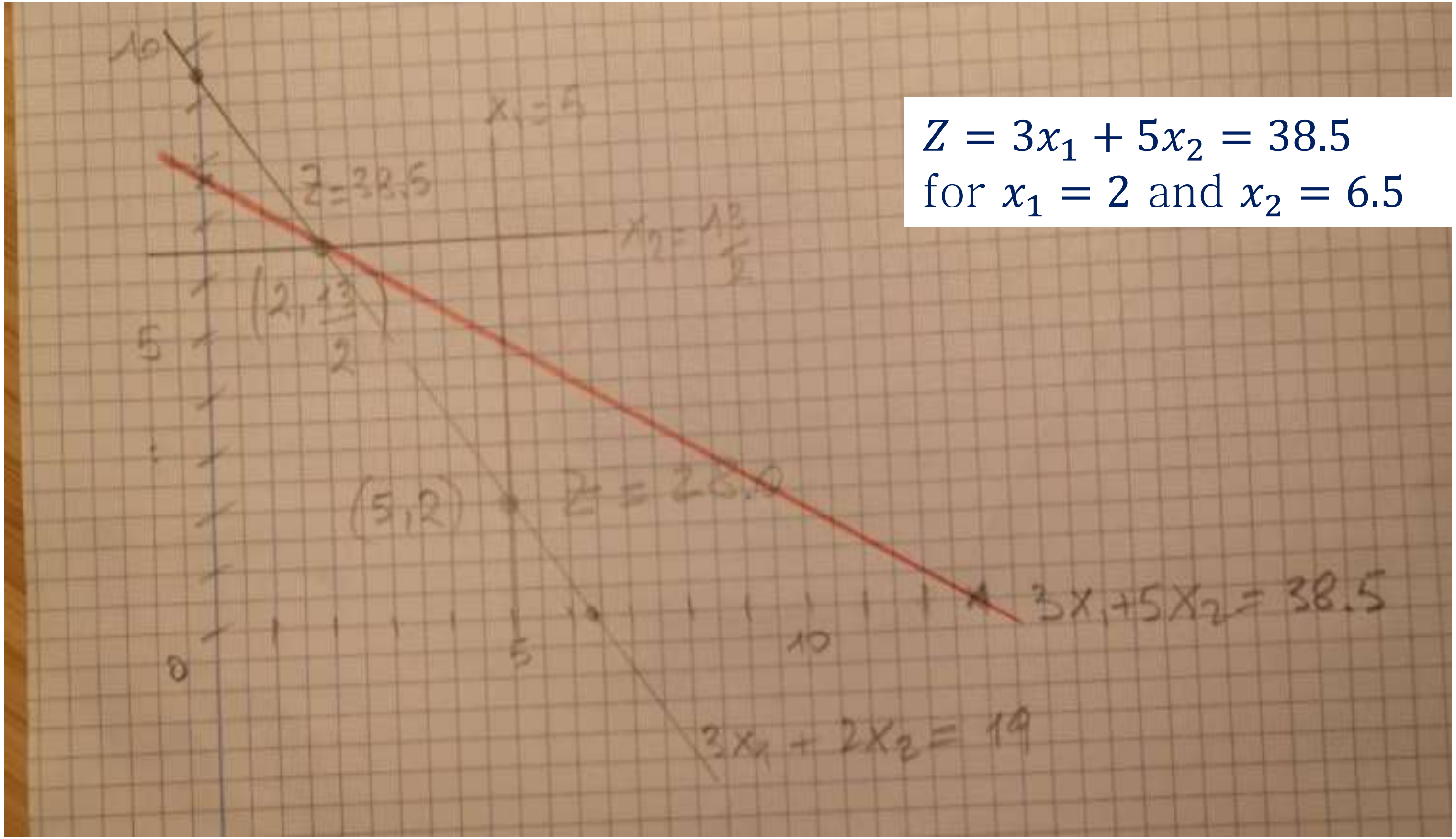
$Z = 3x_1 + 5x_2 = 38.5$
for $x_1 = 2$ and $x_2 = 6.5$



Homework

3. The following table gives a two-way classification of all basketball players at a state university who began their college careers between 2001 and 2005, based on gender and whether or not they graduated. If one of these players is selected at random, find the following probabilities:
   P(female) =165/346
   P(male) =181/346
   P(graduated) =259/346
   P(non graduated) =87/346
   P(female and graduated)
   =P(female∩)graduated =133/346
   P(male and did not graduate)
   =P(male ∩ did not graduate) =55/346

	Graduated	Did Not Graduate	
Male	126	55	181
Female	133	32	165
	259	87	346

Also find P(graduated and did not graduate) = P(graduated ∩ did not graduate). Is this probability zero? If yes, why? Yes, because mutually exclusive

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Homework

4. Using the results achieved in the preceding exercise compute

$P(\text{female or did not graduate}) = P(\text{female} \cup \text{did not graduate}) =$
$= P(\text{female}) + P(\text{did not graduate}) - P(\text{female} \cap \text{did not graduate})$

$P(\text{graduated or male}) = P(\text{graduated} \cup \text{male}) =$
$= P(\text{graduated}) + P(\text{male}) - P(\text{graduated} \cap \text{male})$

Homework

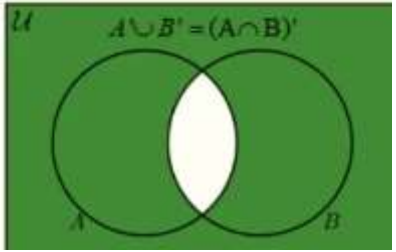
5. A certain state's auto license plates have three letters of the alphabet followed by a three-digit number.
   a. How many different license plates are possible if all three-letter sequences (letter can be repeated) are permitted and any number from 000 to 999 is allowed?

26 letter English alphabet ➔ 26^3 permutations from AAA to ZZZ
1000 numbers allowed
Total= $1000 * 26^3 = 17{,}576{,}000$

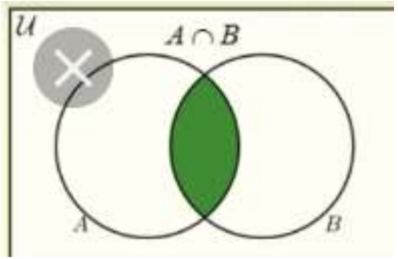
   b) If a witness of a hit-and-run accident says that the first letter on the license plate of the offender's car was a B, that the second letter was an O or a Q, and that the last number was a 5, how many of this state's license plates fit this description?

All groups letter with B and either a O or a Q make $26 + 26$; at possible permutations ending with 5 are 100 (from 00 to 99); total=$52 * 100 = 5{,}200$

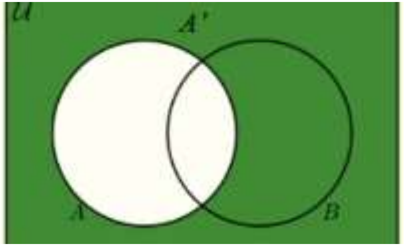
5. Show graphically that this is true



The complement to this is the above



The union of this plus the corresponding plot for B' is the above



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The nut-mix problem: OK the solution killing mixture D, so both solutions legitimate

Table 1

Mixture	Specifications	Selling price: ¢/pound
A	Not less than 50% cashews Not more than 25% peanuts	50
B	Not less than 25% cashews Not more than 50% peanuts	35
D	No specifications	25

Hazels
(avellana)

Source: https://www.woodlandtrust.org.uk/

Peanuts
(maní)

https://www.nutsforlife.com.au

Cashew
(anacardi)

Source: https://www.cashews.org

Table 2

Inputs	Capacity: pounds/day	Price: ¢/pound
C	100	65
H	60	35
P	100	25
Total	260	

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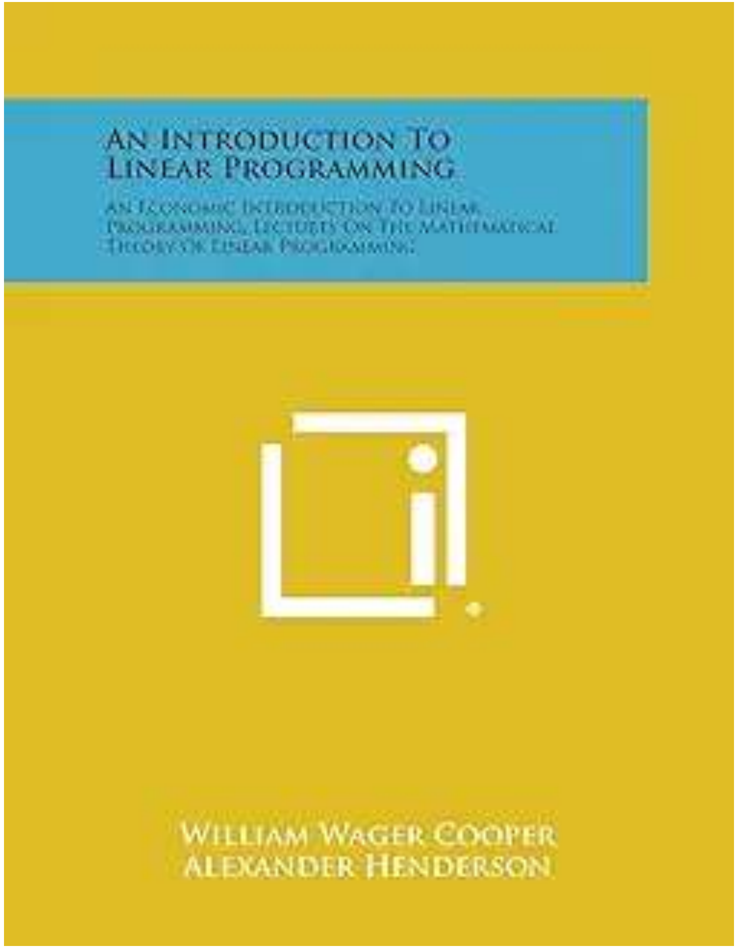
1) Anyone wants to try it on SOLVER?

2) Help to get the source? On US Amazon at $23 but does not ship to Spain

An Introduction to Linear Programming, Abraham Charnes,
William W. Cooper, A. Henderson, John Wiley & Sons, New York



Why is all this important? Fishing expeditions and forking paths …

From Lesson #3

Exploring with global sensitivity analysis

Jorge Luis Borges
(1899–1986)

Taking different storylines within the same novel like Ts'ui Pên

The garden of forking paths: Why multiple comparisons can be a problem, even when there is no "fishing expedition" or "p-hacking" and the research hypothesis was posited ahead of time*

Andrew Gelman† and Eric Loken‡

14 Nov 2013

The garden of forking paths: Why multiple comparisons can be a problem, even when there is no "fishing expedition" or "p-hacking" and the research hypothesis was posited ahead of time*

Andrew Gelman† and Eric Loken‡

14 Nov 2013

'Fishing expedition'? An analyst changing the question asked from the data to squeeze out a publishable effect

'P-hacking'? an analyst torturing the data to pass a significance test (remember the tea drinker of Lesson 1)

The garden of forking paths: Why multiple comparisons can be a problem, even when there is no “fishing expedition” or “p-hacking” and the research hypothesis was posited ahead of time*

Andrew Gelman† and Eric Loken‡

14 Nov 2013

Why this matters?

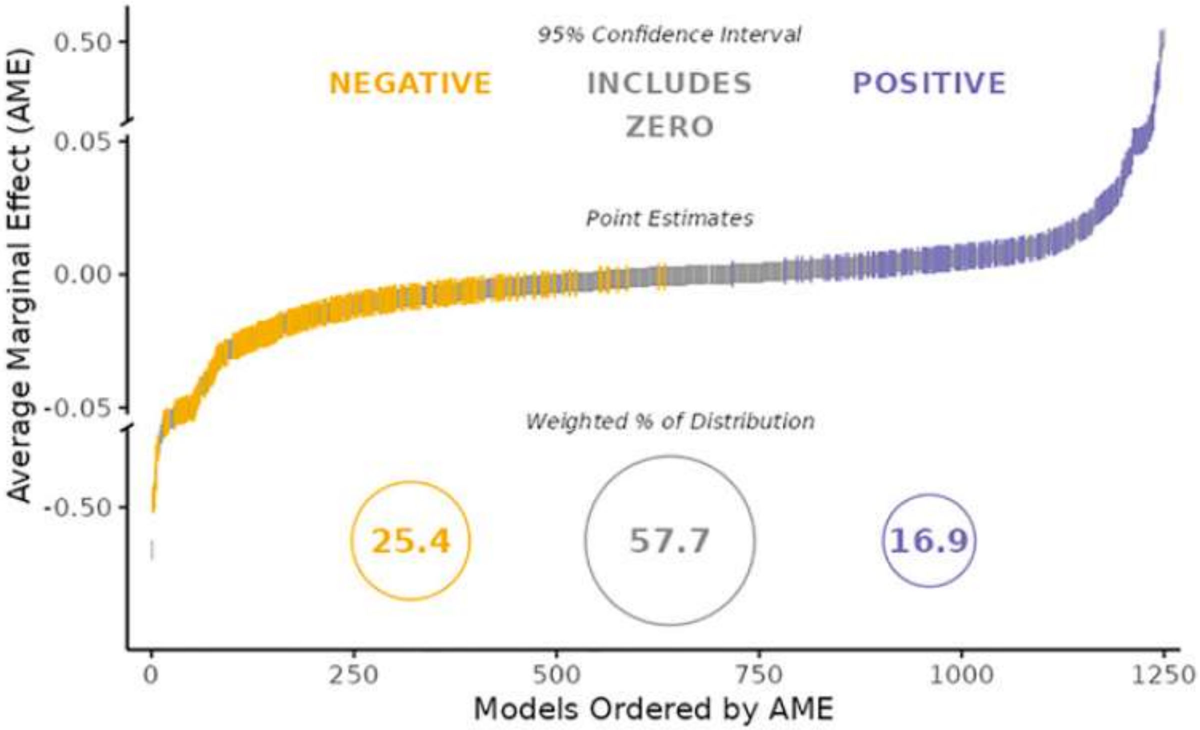
PNAS

RESEARCH ARTICLE | SOCIAL SCIENCES

OPEN

Observing many researchers using the same data and hypothesis reveals a hidden universe of uncertainty

Edited by Douglas Massey, Princeton University, Princeton, NJ; received March 6, 2022; accepted August 22, 2022



"Will different researchers [73 teams] converge on similar findings when analyzing the same data?

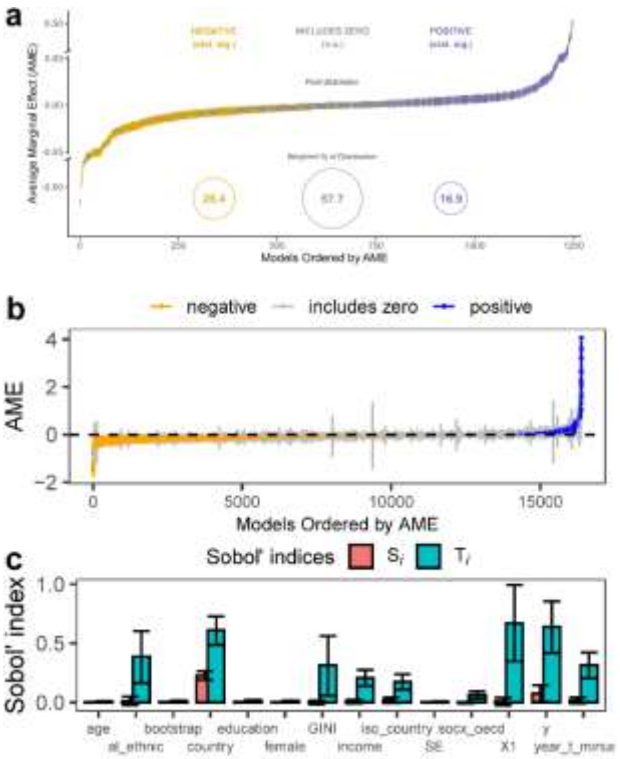
…teams' results varied greatly, ranging from large negative to large positive effects"
(Massey et al. 2022)

Global sensitivity analysis can chart the garden before you enter into it …

What the analyst did in the garden

A simulation of walking in the garden

What sensitivity analysis says about the choices



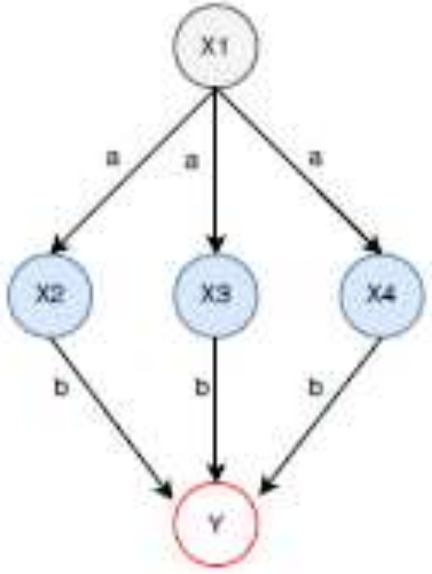
Andrea Saltelli, Arnald Puy, Alessio Lachi, and Nate Breznau, 2024, Global sensitivity analysis unveils the hidden universe of uncertainty in multiverse studies, MetaArXiv Preprints, https://osf.io/preprints/metaarxiv/b67w9.

Another use for uncertainty and sensitivity analysis

In machine learning, check that the algorithm is ‘fair’

PROTECTED ATTRIBUTES:
- Age
- Disability
- National Origin
- Race/color
- Religion
- Sex
- (From the US Equal Opportunity Employment Commission)

Ascertain that an algorithm does not make implicit use of protected attributes (for example in the graph Y must not depend upon x_1)

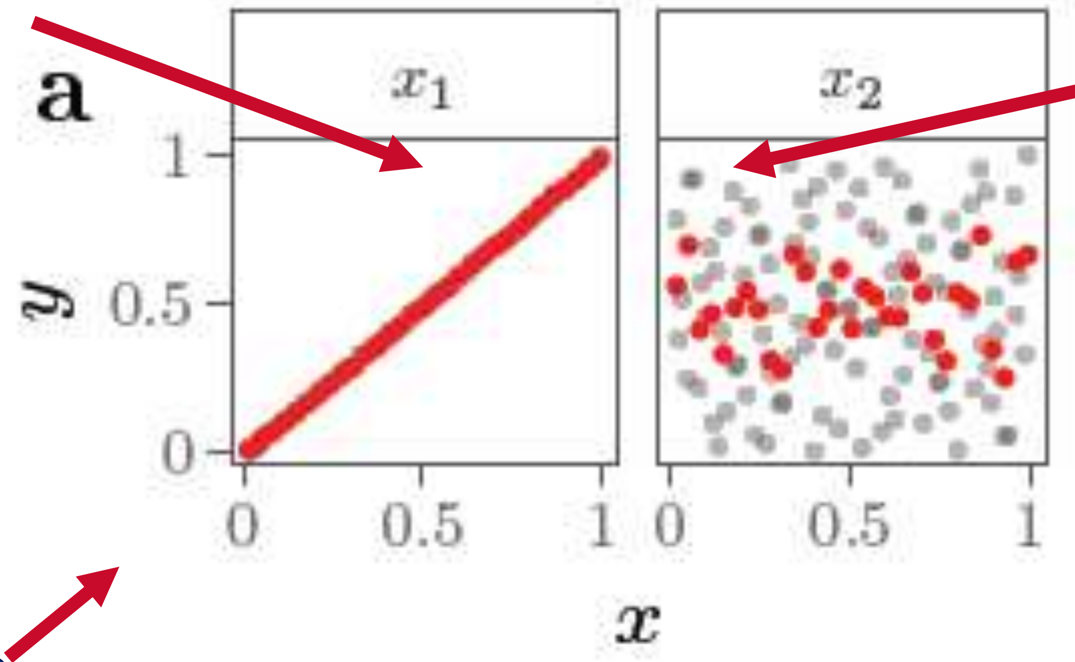


For Machine Learning students: Bénesse, C., Gamboa, F., Loubes, J.-M., & Boissin, T. (2022). Fairness seen as Global Sensitivity Analysis. *Machine Learning*. https://doi.org/10.1007/s10994-022-06202-y

Recap from previous lesson: recognizing variable importance from scatterplots in the x_i, y plane, both in [0,1]

More 'shape'

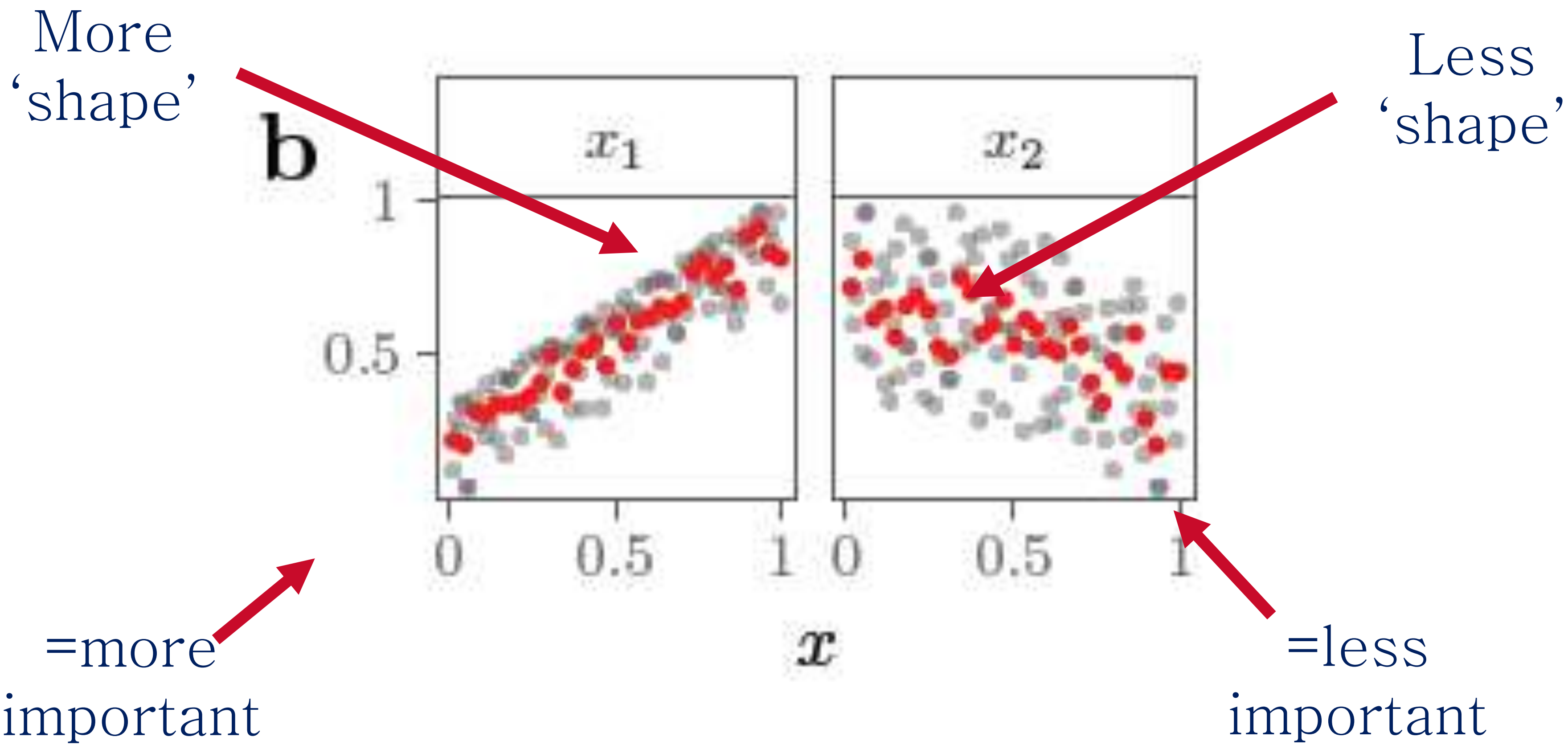
Less 'shape'



=more important

=less important

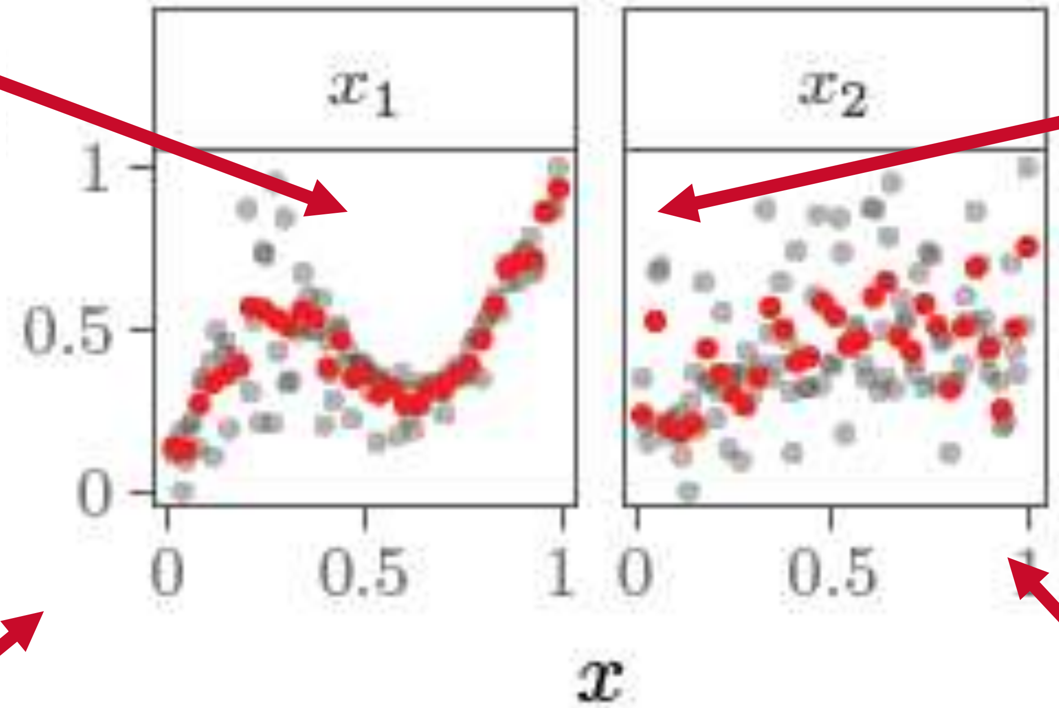
Recap from previous lesson: recognizing variable importance from scatterplots in the x_i, y plane, both in [0,1]



Recap from previous lesson: recognizing variable importance from scatterplots in the x_i, y plane, both in [0,1]

Bigger 'holes'

Smaller 'holes'



=more important

=less important

Just a reminder from previous lessons and a note

	Permutations n elements in classes of k (~~variations~~)	Combinations n elements in classes of k
No repetition	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$
Repetition	n^k	$\binom{n+k-1}{k}$

Somewhere you see $n!$ called a permutation and $\frac{n!}{(n-k)!}$ a variation;

MOST WORKS call these latter permutation in classes of k

Combinations and permutations with repetition: example **three objects ABC in groups of 2**

	Permutations 3 elements in classes of 2 (~~variations~~)	**Combinations 3 elements in classes of 2**
No repetition	AB,BA,AC,CA,BC,CB (3!/1!=6)	AB,AC,BC (3!/2!=3)
Repetition	AA,BB,CC,AB,BA,AC, CA,BC,CB (3^2=9)	AA,BB,CC,AB,AC,BC (4!/(2!2!)=6)

In this set of slides:

12 The Transportation Problem

13 The Assignment Problems (sketched)

14 Network Optimization Models

15 Integer Programming (beginning)

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12.

The Transportation problem

Framing of the problem, assumptions and properties of the solution. Hillier 2014, chapter 9.

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A prototype example of a Transportation Problem: shipping canned peas from canneries to warehouses

Three canneries and four warehouses

Source: Wikipedia Commons

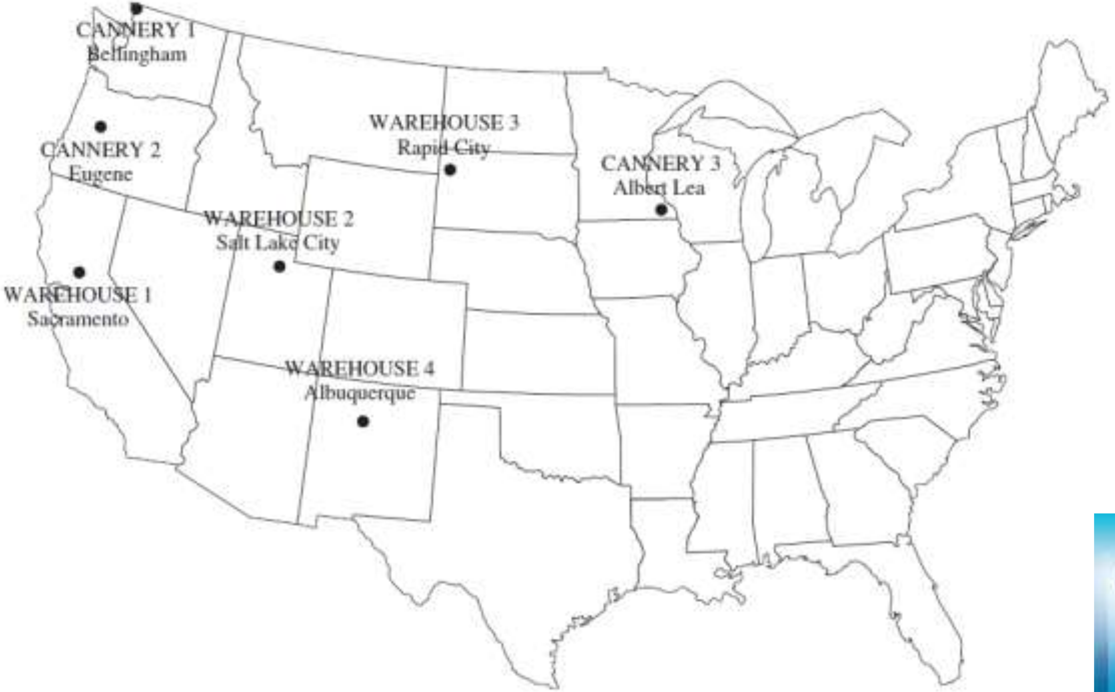

FIGURE 9.1
Location of canneries and warehouses for the P & T Co. problem.

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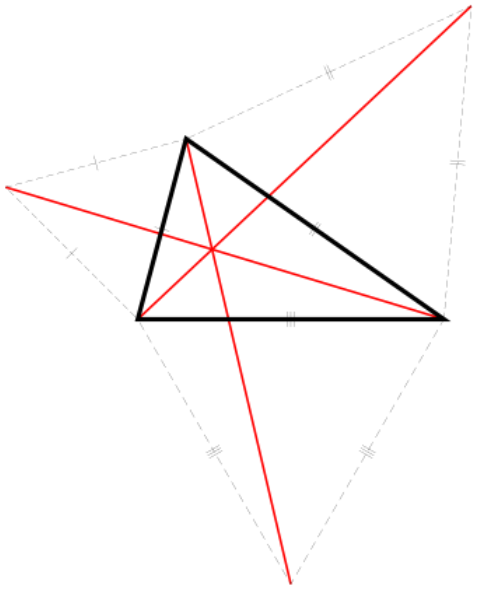
An old type of problem, recall the Torricelli and Fermat point



Source: Wikipedia Commons

1.Construct an equilateral triangle on each of the sides
2. From each of the farmost vertex draw a line the opposite vertex of the original triangle.
3. Where the three lines intersect is the Torricelli-Fermat point.

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A prototype example: shipping canned peas from canneries to warehouses; this table contains all the information; where are the geographical distances?

TABLE 9.2 Shipping data for P & T Co.

	Shipping Cost ($) per Truckload				
	Warehouse				
	1	**2**	**3**	**4**	**Output**
Cannery 1	464	513	654	867	75
Cannery 2	352	416	690	791	125
Cannery 3	995	682	388	685	100
Allocation	80	65	70	85	

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In linear programming the geography can be made to disappear

Here it is replaced by costs per truckload per season

TABLE 9.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	**2**	**3**	**4**	**Output**
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

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A prototype example: shipping canned peas from canneries to warehouses

We know how much moving truckloads costs

■ **TABLE 9.2** Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	**2**	**3**	**4**	**Output**
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

… and how much each warehouse should be provided with

Subject to cannery constraints

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TABLE 9.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

Minimize or maximize? → Minimize

What? → Total shipping cost; decision variable $x_{i,j}, i = 1,2,3; j = 1,2,3,4$ **number** of truckloads from cannery i to warehouse j

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TABLE 9.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

$$\text{Minimize total shipping cost } Z = 464\, x_{1,1} + 513\, x_{1,2} + 654\, x_{1,3} + 867\, x_{1,4} + 352\, x_{2,1} + 416\, x_{2,2} + 690\, x_{2,3} + 791\, x_{2,4} + 995 x_{3,1} + 682\, x_{3,2} + 388\, x_{3,3} + 685\, x_{3,4}$$

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TABLE 9.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
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	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

Subject to cannery constraints

$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 75$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 125$$
$$x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = 100$$

and warehouse constrains

$$x_{1,1} + x_{2,1} + x_{3,1} = 80$$
$$x_{1,2} + x_{2,2} + x_{3,2} = 65$$
$$x_{1,3} + x_{2,3} + x_{3,3} = 70$$
$$x_{1,4} + x_{2,4} + x_{3,4} = 85$$

$x_{i,j} \geq 0$ $(i = 1,2,3; j = 1,2,3,4)$; truckload from cannery $i = 1,2,3$ to warehouse $j = 1,2,3,4$

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■ **TABLE 9.2** Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

Anything noticeable about these two sets of numbers?

Supply and demand balance out at 300

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■ **TABLE 9.2** Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	**2**	**3**	**4**	**Output**
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

12 decision variables

Why?

(3 ways to choose a cannery and 4 ways to choose a warehouse)

$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 75$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 125$$
$$x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = 100$$

3 cannery constraints⋯

$$x_{1,1} + x_{2,1} + x_{3,1} = 80$$
$$x_{1,2} + x_{2,2} + x_{3,2} = 65$$
$$x_{1,3} + x_{2,3} + x_{3,3} = 70$$
$$x_{1,4} + x_{2,4} + x_{3,4} = 85$$

⋯+ 4 warehouse constraints = seven constraints in total

What is new here with respect to our previous LP problems?

TABLE 3.1 Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

These don't

TABLE 9.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	300

These balance out

What else is new here with respect to our previous LP problems?

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

Constraint horizontally

TABLE 9.2 Shipping data for P & T Co.

	Shipping Cost ($) per Truckload — Warehouse 1	2	3	4	Output
Cannery 1	464	513	654	867	75
2	352	416	690	791	125
3	995	682	388	685	100
Allocation	80	65	70	85	300

Constraint horizontally and vertically

Else?

■ **TABLE 3.1** Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m,$$

■ **TABLE 9.2** Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	300

Functional constraint equations coefficients a_{ij} are all $= 1$

$$x_{1,1} + x_{2,1} + x_{3,1} = 80$$
$$x_{1,2} + x_{2,2} + x_{3,2} = 65$$
$$x_{1,3} + x_{2,3} + x_{3,3} = 70$$
$$x_{1,4} + x_{2,4} + x_{3,4} = 85$$

$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 75$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 125$$
$$x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = 100$$

More?

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

$$= 464\, x_{1,1} + 513\, x_{1,2} + 654\, x_{1,3} + 867\, x_{1,4} + 352\, x_{2,1} + 416\, x_{2,2} + 690\, x_{2,3} + 791\, x_{2,4} + 995 x_{3,1} + 682\, x_{3,2} + 388\, x_{3,3} + 685\, x_{3,4}$$

TABLE 9.2 Shipping data for P & T Co.

	Shipping Cost ($) per Truckload				
	Warehouse				
	1	2	3	4	Output
Cannery 1	464	513	654	867	75
2	352	416	690	791	125
3	995	682	388	685	100
Allocation	80	65	70	85	300

These go into the optimization equation

The constraints can be written as a distinct pattern that is characteristic of the Transportation and Assignment Problem

Combining the 12 decision variables with the 7 constraints produces this pattern

Decision variables 1 to12 ➔

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &= 75 \\
x_{21} + x_{22} + x_{23} + x_{24} &= 125 \\
x_{31} + x_{32} + x_{33} + x_{34} &= 100 \\
\hline
x_{11} + x_{21} + x_{31} &= 80 \\
x_{12} + x_{22} + x_{32} &= 65 \\
x_{13} + x_{23} + x_{33} &= 70 \\
x_{14} + x_{24} + x_{34} &= 85
\end{aligned}
$$

Cannery

Warehouse

Constraints 1 to 7 ➔

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Decision variables 1 to12 →

Table/matrix form of the constraint coefficients; coefficients equal to zero are left as blanks

$$
\begin{array}{cccccccccccccccccc}
x_{11} & + & x_{12} & + & x_{13} & + & x_{14} & & & & & & & & & & & = 75 \\
& & & & & & & & x_{21} + x_{22} + x_{23} + x_{24} & & & & & & & & & = 125 \\
& & & & & & & & & & & & & & x_{31} + x_{32} + x_{33} + x_{34} & & & = 100 \\
x_{11} & & & & & & & + & x_{21} & & & + & x_{31} & & & & & = 80 \\
& & x_{12} & & & & & + & x_{22} & & & + & x_{32} & & & & & = 65 \\
& & & & x_{13} & & & + & x_{23} & & & + & x_{33} & & & & & = 70 \\
& & & & & & x_{14} & + & x_{24} & & & + & x_{34} & & & & & = 85
\end{array}
$$

Cannery

Warehouse

Constraints 1 to 7 ↓

■ **TABLE 3.1** Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	**1**	**2**	**Production Time Available per Week, Hours**
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

In standard linear programming zeros are explicit

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The are in principle 7*12=84 cells in this matrix but most cells are empty

How many cells are not empty?

24 cells non empty. Why?

Each of the two sets of constraints (cannery and warehouse) uses each and every of the 12 decision variable just once

Decision variables 1 to12 ➔

$$
\begin{array}{llllllllllllll}
x_{11} + x_{12} + x_{13} + x_{14} & & & & = 75 \\
& x_{21} + x_{22} + x_{23} + x_{24} & & & = 125 \\
& & x_{31} + x_{32} + x_{33} + x_{34} & & = 100 \\
\hline
x_{11} & + x_{21} & + x_{31} & & = 80 \\
x_{12} & + x_{22} & + x_{32} & & = 65 \\
x_{13} & + x_{23} & + x_{33} & & = 70 \\
x_{14} & + x_{24} & + x_{34} & & = 85
\end{array}
$$

Cannery

Warehouse

Constraints 1 to 7 ➔

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This is the distinct pattern linking decision variables and constraints in the Transportation and Assignment Problem

Decision variables 1 to12 →

Constraints 1 to 7 ↓

$$
\begin{array}{l}
x_{11} + x_{12} + x_{13} + x_{14} = 75 \\
x_{21} + x_{22} + x_{23} + x_{24} = 125 \\
x_{31} + x_{32} + x_{33} + x_{34} = 100 \\
x_{11} + x_{21} + x_{31} = 80 \\
x_{12} + x_{22} + x_{32} = 65 \\
x_{13} + x_{23} + x_{33} = 70 \\
x_{14} + x_{24} + x_{34} = 85
\end{array}
$$

TABLE 9.3 Constraint coefficients for P & T Co.

Coefficient of:

	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	
A =	1	1	1	1									Cannery constraints
					1	1	1	1					
									1	1	1	1	
	1				1				1				Warehouse constraints
		1				1				1			
			1				1				1		
				1				1				1	

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Decision variables 1 to12 →

Constraints 1 to 7 ↓

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &= 75\\
x_{21} + x_{22} + x_{23} + x_{24} &= 125\\
x_{31} + x_{32} + x_{33} + x_{34} &= 100\\
x_{11} + x_{21} + x_{31} &= 80\\
x_{12} + x_{22} + x_{32} &= 65\\
x_{13} + x_{23} + x_{33} &= 70\\
x_{14} + x_{24} + x_{34} &= 85
\end{aligned}
$$

Key characteristic of the **Transportation and Assignment Problems**: the a_{ij} coefficients in the constraints are either zeros (most of them) or one (few), that show a distinctive pattern

TABLE 9.3 Constraint coefficients for P & T Co.

	Coefficient of:												
	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	
A =	1	1	1	1									Cannery constraints
					1	1	1	1					
									1	1	1	1	
	1				1				1				Warehouse constraints
		1				1				1			
			1				1				1		
				1				1				1	

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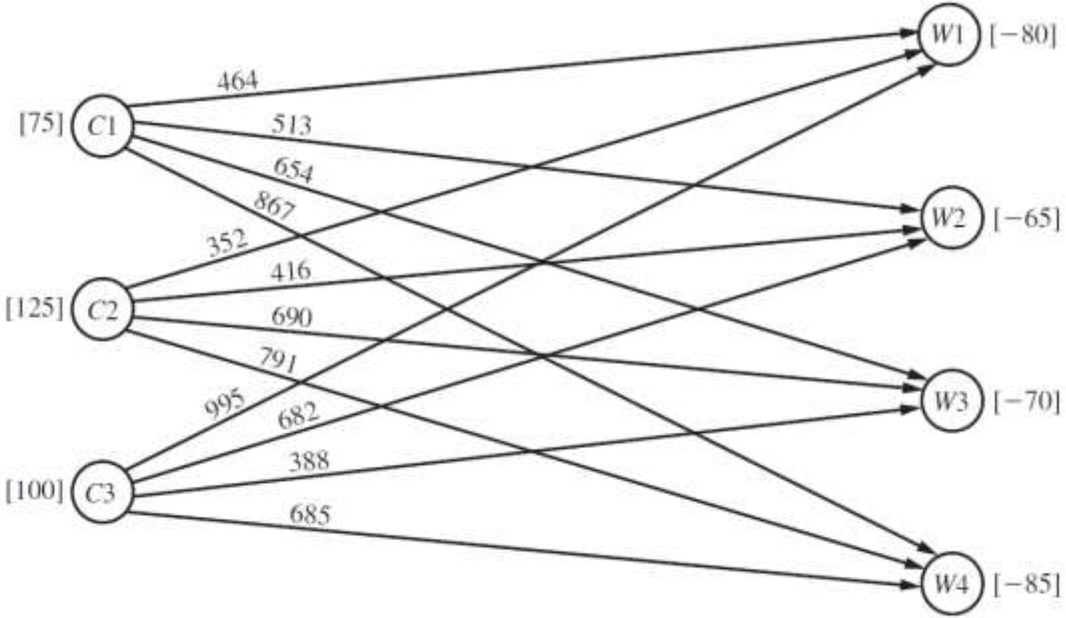
TABLE 9.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

Or as a graph/network representation



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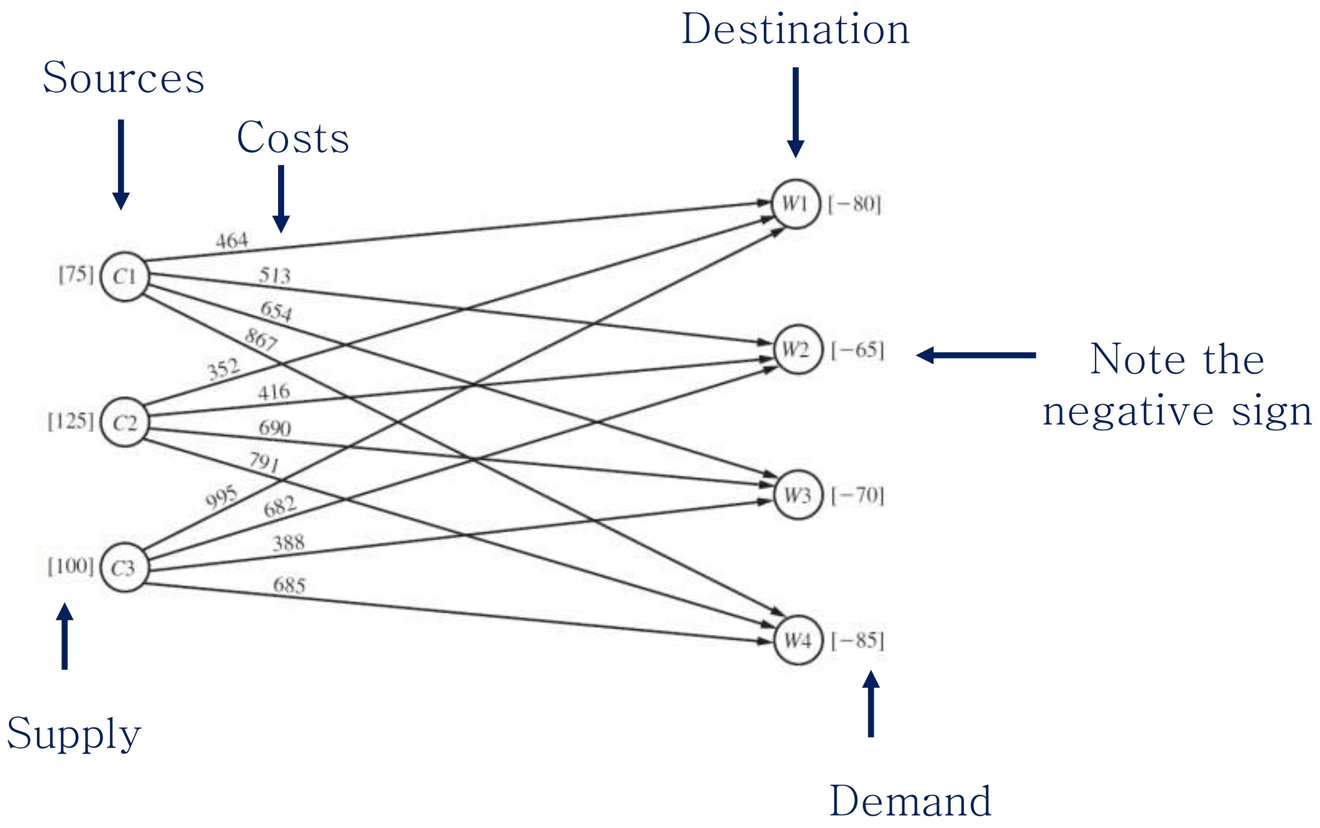
Terminology of the Transportation and Assignment Problem



$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 75$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 125$$
$$x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = 100$$

$$x_{1,1} + x_{2,1} + x_{3,1} = 80$$
$$x_{1,2} + x_{2,2} + x_{3,2} = 65$$
$$x_{1,3} + x_{2,3} + x_{3,3} = 70$$
$$x_{1,4} + x_{2,4} + x_{3,4} = 85$$

The = sign (instead of ≤≥) in the supply and demand represents the **requirement assumption** of the Transportation and Assignment Problem: **supply and demand are fixed**

➔ No wiggle room: the supply must be supplied and the demand must be met in total

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■ **TABLE 9.2** Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

Minimize total shipping cost $Z =$

$$= 464\,x_{1,1} + 513\,x_{1,2} + 654\,x_{1,3} + 867\,x_{1,4}$$
$$+ 352\,x_{2,1} + 416\,x_{2,2} + 690\,x_{2,3} + 791\,x_{2,4}$$
$$+ 995x_{3,1} + 682\,x_{3,2} + 388\,x_{3,3} + 685\,x_{3,4}$$

The **cost assumption**: distributing units from any source to any destination is proportional to the number of units distributed; if c_{ij} is the unit cost and x_{ij} the number of units, the cost is simply $c_{ij}x_{ij}$

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The **requirements assumption** is typic of transportation problem, while the **cost assumption** is not new, right?

What are the assumptions we studied already?

Assumptions of linear programming

Proportionality: The contribution of each activity to the value of the objective function Z is proportional to the level of the activity x_j increase in the objective function Z, as represented by the $c_j x_j$ terms

$$\text{Maximize} \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

Additivity: Every function in a linear programming model (whether the objective function or the function on the left-hand side of a functional constraint) is the sum of the individual contributions of the respective activities

Divisibility: Decision variables in a linear programming model are allowed to have any values, including noninteger values, that satisfy the functional and nonnegativity constraints. Thus, these variables are not restricted to just integer values. Since each decision variable represents the level of some activity, it is being assumed that the activities can be run at fractional levels

When a decision variable **must** be an integer, it becomes a case of integer programming

Certainty: The value assigned to the parameters (the a_{ij}'s, b_i's, and c_j's) of a linear programming model are assumed to be known constants

Whether or not actual transportation is involved, any problem in the format of this table that obeys the requirement and cost assumption is a transportation problem

TABLE 9.5 Parameter table for the transportation problem

		Cost per Unit Distributed				
		Destination				
		1	**2**	⋯	***n***	**Supply**
Source	1	c_{11}	c_{12}	⋯	c_{1n}	s_1
	2	c_{21}	c_{22}	⋯	c_{2n}	s_2
	⋮	⋯	⋯	⋯	⋯	⋮
	m	c_{m1}	c_{m2}	⋯	c_{mn}	s_m
Demand		d_1	d_2	⋯	d_n	

TABLE 9.4 Terminology for the transportation problem

Prototype Example	**General Problem**
Truckloads of canned peas	Units of a commodity
Three canneries	m sources
Four warehouses	n destinations
Output from cannery i	Supply s_i from source i
Allocation to warehouse j	Demand d_j at destination j
Shipping cost per truckload from cannery i to warehouse j	Cost c_{ij} per unit distributed from source i to destination j

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Compact formulation for a problem with m sources s and n destinations d:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to source and demand constraints

$$\sum_{j=1}^{n} x_{ij} = s_i \quad \text{for} \quad i = 1,2, \dots m$$

$$\sum_{i=1}^{m} x_{ij} = d_j \text{ for } j = 1,2, \dots n$$

$$x_{ij} \geq 0 \text{ for } (i = 1,2, \dots m;\ j = 1,2, \dots n)$$

TABLE 9.5 Parameter table for the transportation problem

		Cost per Unit Distributed				
		Destination				
		1	**2**	···	***n***	**Supply**
Source	1	c_{11}	c_{12}	···	c_{1n}	s_1
	2	c_{21}	c_{22}	···	c_{2n}	s_2
	⋮					⋮
	m	c_{m1}	c_{m2}	···	c_{mn}	s_m
Demand		d_1	d_2	···	d_n	

BUT a transportation problem will have feasible solution if and only if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

(supply and demand balance out as in the example)

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Compact formulation for a problem with m sources s and n destinations d:

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to

$\sum_{i=1}^{m} x_{ij} = d_j$ for $j = 1,2, \dots n$

$\sum_{j=1}^{n} x_{ij} = s_i$ for $i = 1,2, \dots m$

$x_{ij} \geq 0$ for $(i = 1,2, \dots m;\ j = 1,2, \dots n)$

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

(supply and demand balance out)

The integer solutions property: For transportation problems where every s_i and d_i have an integer value, all basic feasible (BF) solutions (including an optimal one) also have integer values

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TABLE 9.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

Optimal solution with Excel Solver

0	20	0	55
80	45	0	0
0	0	70	30

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Tom would like 2 pints of home brew today and an additional 7 pints of home brew tomorrow. Dick is willing to sell a maximum of 5 pints total at a price of $3.00 per pint today and $2.70 per pint tomorrow. Harry is willing tc sell a maximum of 4 pints total at a price of $2.90 per pint today and $2.80 per pint tomorrow. Tom wishes to know what his purchases should be to minimize his cost while satisfying his thirst requirements. What would you do being Tom?

Formulate this problem as a *transportation problem* by constructing the appropriate parameter table; verify the requirement assumption and solve

	Today	Tomorrow	
Dick	3.	2.70	5
Harry	2.90	2.80	4
Tom/day	2	7	

Now write the equations

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Tom would like 2 pints of home brew today and an additional 7 pints of home brew tomorrow. Dick is willing to sell a maximum of 5 pints total at a price of $3.00 per pint today and $2.70 per pint tomorrow. Harry is willing to sell a maximum of 4 pints total at a price of $2.90 per pint today and $2.80 per pint tomorrow. Tom wishes to know what his purchases should be to minimize his cost while satisfying his thirst requirements.

Formulate this problem as a *transportation problem* by constructing the appropriate parameter table; verify the requirement assumption and solve

	Today	Tomorrow	
Dick	3.	2.70	5
Harry	2.90	2.80	4
Tom/day	2	7	

Minimize

$3x_{11} + 2.7x_{12} + 2.9x_{21} + 2.8x_{22}$

Subject to

$x_{11} + x_{12} \leq 5$

$x_{21} + x_{22} \leq 4$

Note the deviation from the standard transportation problem, ≤ instead of =

and

$x_{11} + x_{21} = 2$

$x_{12} + x_{22} = 7$

That is soluble because $4 + 5 = 2 + 7 = 9$

	Today	Tomorrow	
Dick	3.	2.70	5
Harry	2.90	2.80	4
Tom/day	2	7	

Minimize
$3x_{11} + 2.7x_{12} + 2.9x_{21} + 2.8x_{22}$

Subject to
$x_{11} + x_{12} \leq 5$
$x_{21} + x_{22} \leq 4$
and
$x_{11} + x_{21} = 2$
$x_{12} + x_{22} = 7$

The solution is

$$x_{11} = 0$$
$$x_{12} = 5$$
$$x_{21} = 2$$
$$x_{22} = 2$$

That gives $Z = 2.7 * 5 + 2.9 * 2 + 2.80 * 2 = 24.9$

In some real problems, the supplies actually represent *maximum* amounts (rather than fixed amounts) to be distributed. Similarly, in other cases, the demands represent maximum amounts (rather than fixed amounts) to be received. Such problems do not quite fit the model for a transportation problem because they violate the *requirements assumption* …which is?

No wiggle room! All supply out and all demand in

However, soon we see how to *reformulate* the problem to get around this constraint

13.

The Assignment problem

A brief sketch. Hillier 2014, chapter 9.

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The assignment problem is a special type of linear programming problem where **assignees** are being assigned to perform **tasks**

Charles Chaplin's Modern Times, source http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

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1. The number of assignees and the number of tasks are the same.
2. Each assignee is to be assigned to exactly one task.
3. Each task is to be performed by exactly one assignee.
4. There is a cost c_{ij} associated with assignee i, $(i = 1,2, \ldots n)$ performing task j, $(j = 1,2, \ldots n)$.
5. The objective is to determine how all n assignments should be made to minimize the total cost ⋯ but

Source: Wikipedia Commons

Charles Chaplin’s Modern Times, source
http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

In fact, the assignment problem is just a special type of transportation problem where the **sources now are assignees** and the **destinations now are tasks** and where:

Number of sources **m** = number of destinations **n**,
Every supply $s_i = 1$,
Every demand $d_j = 1$

Number of sources m = number of destinations n,
Every supply $s_i = 1$,
Every demand $d_i = 1$

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1,2, \dots n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for} \quad i = 1,2, \dots n$$

$$x_{ij} \geq 0 \text{ for } (i = 1,2, \dots n;\ j = 1,2, \dots n)$$

Plus

x_{ij}= binary (0 or 1) for ($i = 1,2, \dots n;\ j = 1,2, \dots n$)

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Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to

$\sum_{i=1}^{n} x_{ij} = 1$ for $j = 1,2, \dots n$ ⟵ Each task must be served

$\sum_{j=1}^{n} x_{ij} = 1$ for $i = 1,2, \dots n$ ⟵ Each assignee must have work

$x_{ij} \geq 0$ for $(i = 1,2, \dots n;\ j = 1,2, \dots n)$

Plus

x_{ij} = binary $(0\ or\ 1)$ for

$(i = 1,2, \dots n;\ j = 1,2, \dots n)$

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Thus assignment and transportation share the same useful properties in terms of existence of integer solutions

Source: Wikipedia Commons

Charles Chaplin's Modern Times, source
http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

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Assignment and transportation have same network representation

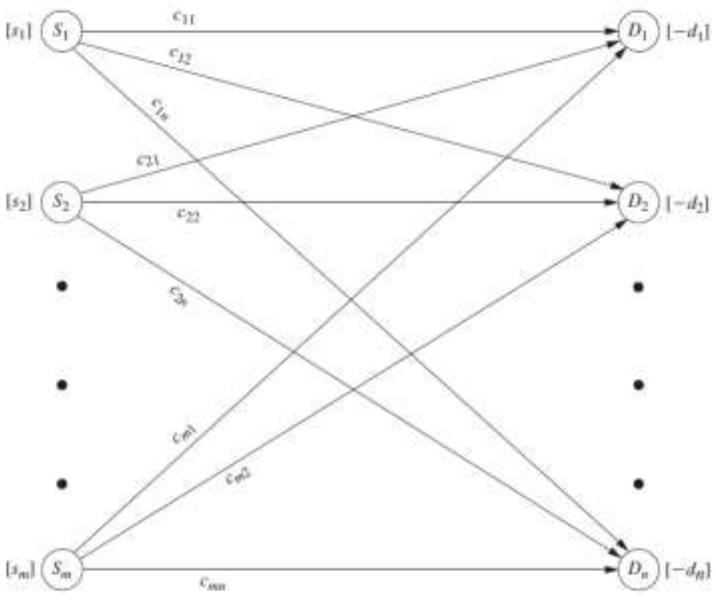

FIGURE 9.3
Network representation of the transportation problem.

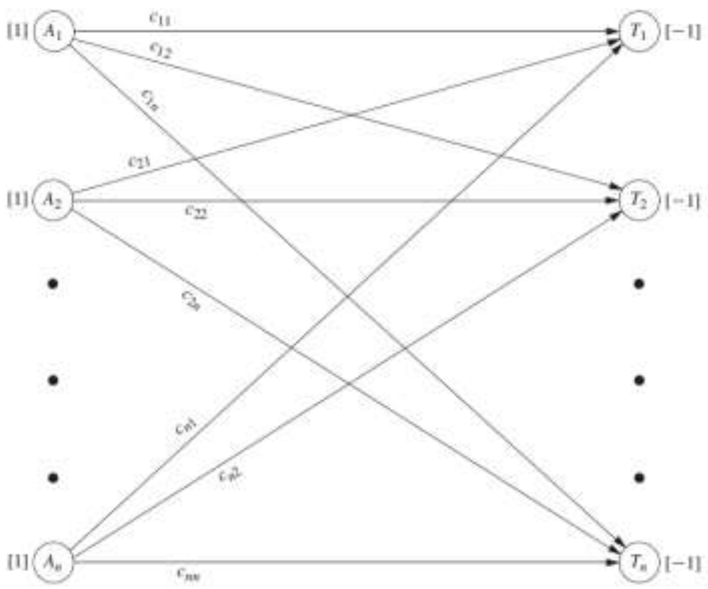

FIGURE 9.5
Network representation of the assignment problem.

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A typical problem offered in the book locating three machine among four facilities, with different cost per machine / facility

TABLE 9.24 Materials-handling cost data ($) for Job Shop Co.

		Location			
		1	**2**	**3**	**4**
Machine	1	13	16	12	11
	2	15	—	13	20
	3	5	7	10	6

		Task (Location)			
		1	**2**	**3**	**4**
Assignee (Machine)	1	13	16	12	11
	2	15	*M*	13	20
	3	5	7	10	6

Machine 2 cannot go to location 2, so a very large cost M in entered in the empty cell

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But …

TABLE 9.24 Materials-handling cost data ($) for Job Shop Co.

		Location			
		1	2	3	4
Machine	1	13	16	12	11
	2	15	—	13	20
	3	5	7	10	6

TABLE 9.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	2	3	4
Assignee (Machine)	1	13	16	12	11
	2	15	*M*	13	20
	3	5	7	10	6
→	4(*D*)	0	0	0	0

Violates the requirement assumption: **No wiggle room:** the supply must be supplied and the demand must be met in total

TABLE 9.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

Transportation

Note: in the transportation problems one can have e.g. 3 canneries and 4 warehouses but in the assignment problems number of sources and number of destination must be equal

TABLE 9.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	2	3	4
	1	13	16	12	11
Assignee	2	15	*M*	13	20
(Machine)	3	5	7	10	6
	4(*D*)	0	0	0	0

Transportation

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In some real problems, the supplies actually represent *maximum* amounts (rather than fixed amounts) to be distributed. Similarly, in other cases, the demands represent maximum amounts (rather than fixed amounts) to be received. Such problems do not quite fit the model for a transportation problem because they violate the *requirements assumption*
...which is?

No wiggle room! All supply out and all demand met

However, soon we see how to *reformulate* the problem to get around this constraint

reformulate =
using dummies

TABLE 9.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	**2**	**3**	**4**
Assignee (Machine)	1	13	16	12	11
	2	15	*M*	13	20
	3	5	7	10	6
	4(*D*)	0	0	0	0

Since assignees and tasks must be equal a dummy machine is introduced

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A typical problem offered in the book locating three machine among four facilities, with different cost per machine / facility

TABLE 9.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	2	3	4
	1	13	16	12	11
Assignee	2	15	*M*	13	20
(Machine)	3	5	7	10	6
	4(*D*)	0	0	0	0

Can you guess the solution "by inspection?"

Machine 1 to location 4
Machine 2 to location 3
Machine 3 to location 1
The algorithms (not described here) would assign the dummy machine 4 to location 2

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14.

Network Optimization Models

More network problems: shortest-path problem, the minimum spanning tree problem, maximum flow problem. Hiller 2014, chapter 10.

Many **network optimization models** are special types of linear programming problems – e.g. the **transportation problem and the assignment problem**

… which indeed have a network representation

Assignment and transportation have same network representation

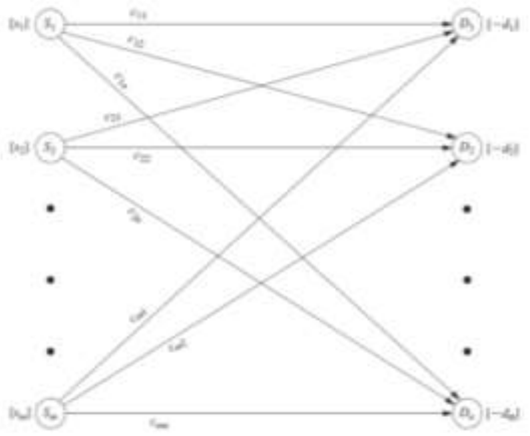

FIGURE 9.3
Network representation of the transportation problem.

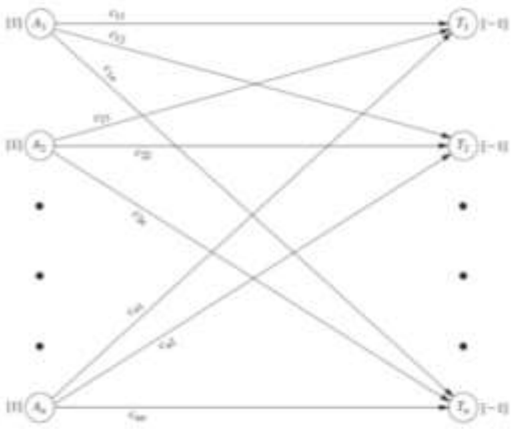

FIGURE 9.5
Network representation of the assignment problem.

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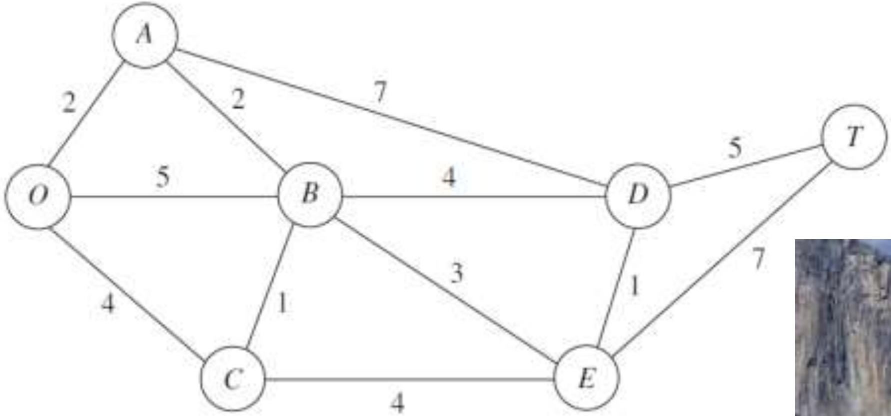
Our new prototype problem – the “Seervada Park” road system



■ **FIGURE 10.1**
The road system for Seervada Park.

Source: https://www.klook.com/en-US/activity/28218-yosemite-park-giant-sequoia-day-tour-san-francisco/?

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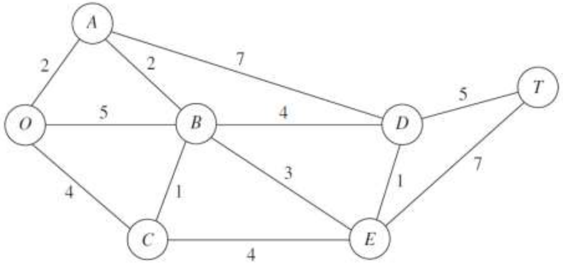
Three practical problems

- **Shortest path** from entrance O to scenic point T
- Minimum length of telephone lines covering all tracks (**minimum spanning tree**)
- **Maximum flow** of mini-trains carrying non trekkers from entrance O to scenic point T



Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

Source: https://www.klook.com/en-US/activity/28218-yosemite-park-giant-sequoia-day-tour-san-francisco/?

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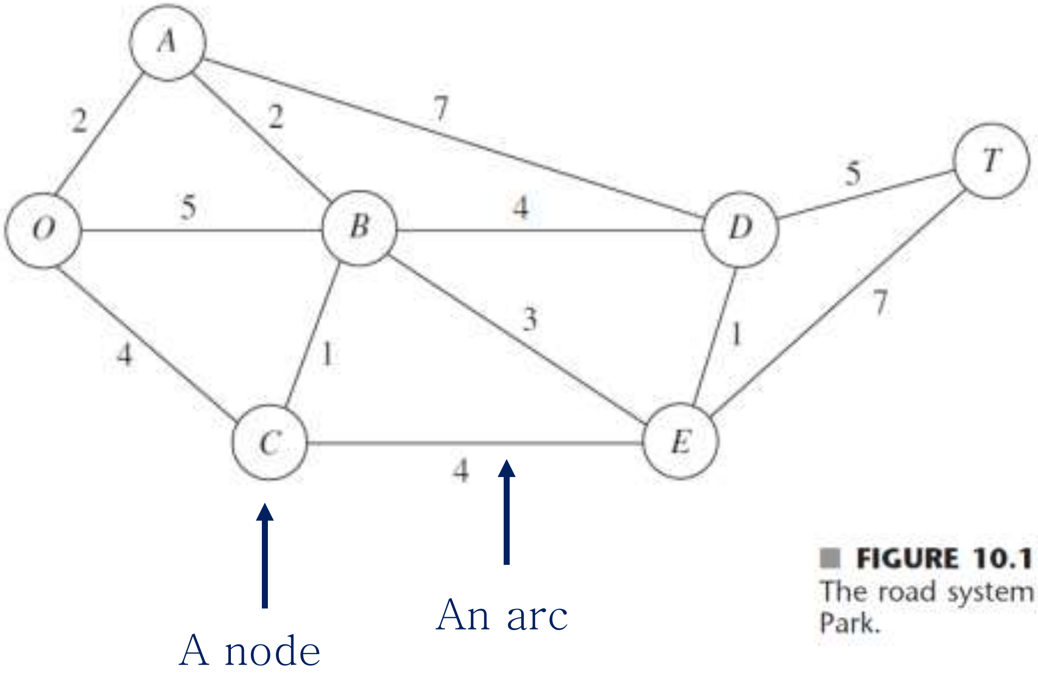
Some terminology: nodes (or vertices), arcs (or links or edges or branches)



■ **FIGURE 10.1**
The road system for Seervada Park.

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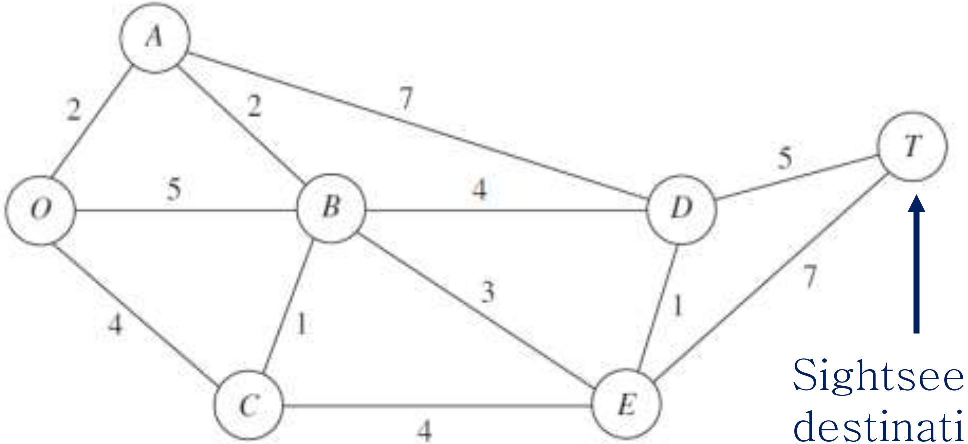

Entrance,
origin

Sightseeing,
destination

The trains trough the park represent a type of 'flow' through the arcs

Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

TABLE 10.1 Components of typical networks

Nodes	Arcs	Flow
Intersections	Roads	Vehicles
Airports	Air lanes	Aircraft
Switching points	Wires, channels	Messages
Pumping stations	Pipes	Fluids
Work centers	Materials-handling routes	Jobs

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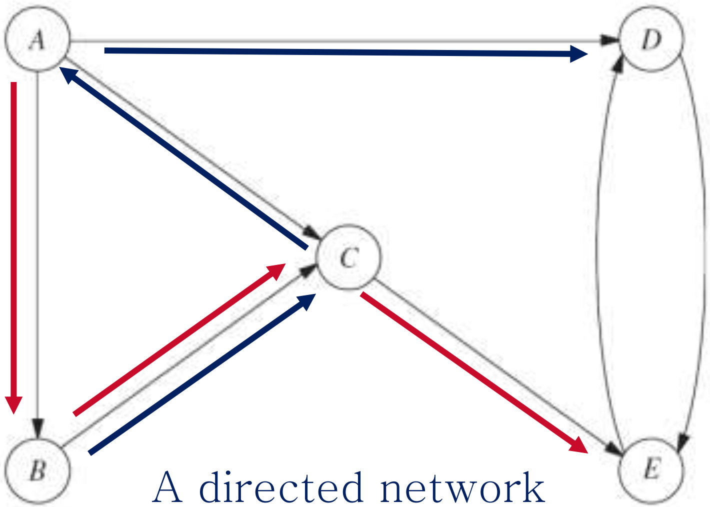
More terminology:

Directed **arcs** (flow only in one directions) and undirected arcs or link (flow in both directions)
Networks can also be directed (only directed arcs) or undirected
A **path** trough nodes can be directed when every step from node i to node j is in the direction of j.

A→B →C→E = directed path

B→C→A→D = undirected path

If you think of this as a flow in a one-way pipe, this is impossible, but it will come handy later



A directed network

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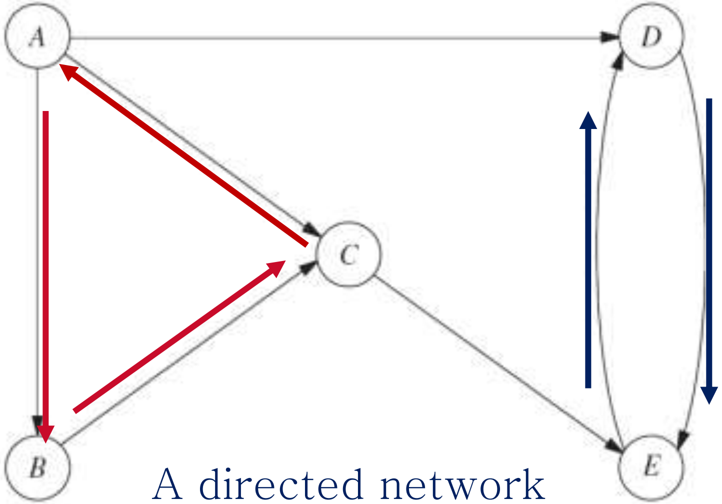
Cycles can also be directed or undirected

D→E →D = directed cycle

A→B→C→A = undirected cycle



A directed network

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Note that our links have no arrows, the network is made of undirected arcs. It is thus…

… and undirected network

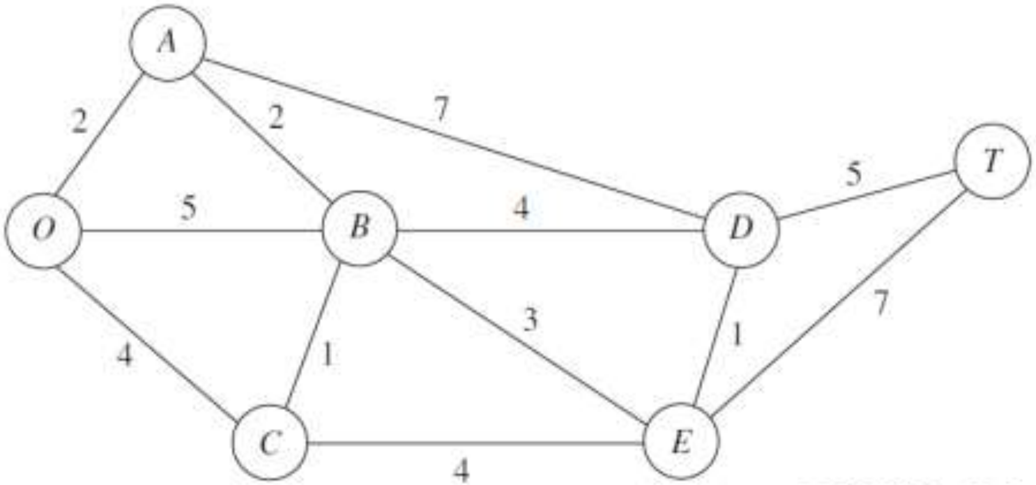

FIGURE 10.1
The road system for Seervada Park.

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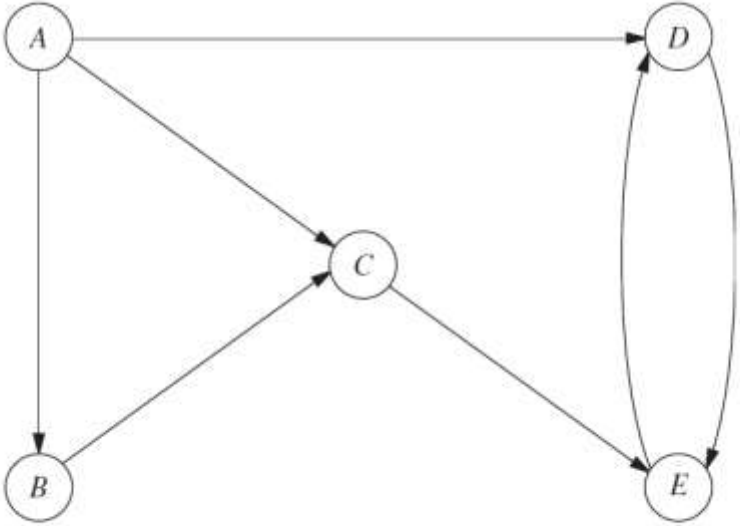
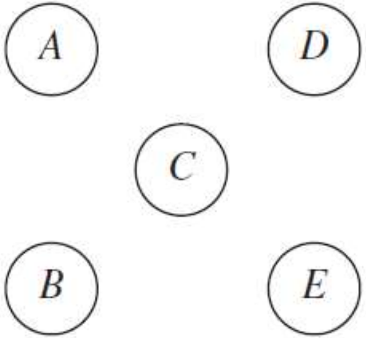
More terminology: starting from bare nodes, **trees** can be grown

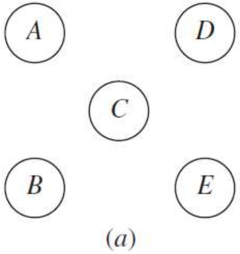


A network;
stripping the
arc one gets ⋯

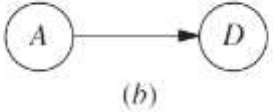


⋯ bare nodes

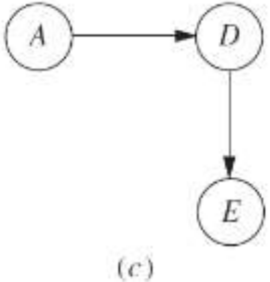
Starting from bare nodes, **trees** can be grown



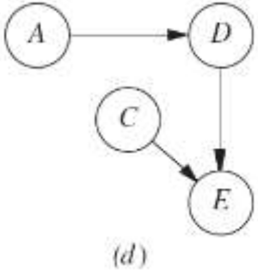
(a) bare nodes



(b) Tree with one arc

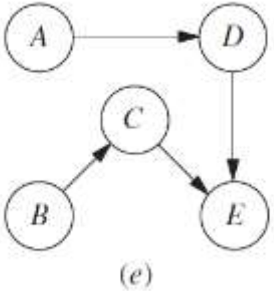


(c) Tree with two arcs



(d) Tree with three arcs

(e) Spanning tree: all nodes connected by directed arcs



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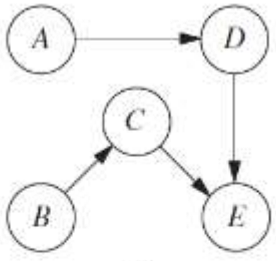
(e) Spanning tree: all nodes connected by directed arcs

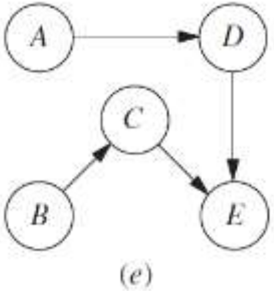


(e)

A spanning tree connects ***n*** nodes with ***n-1*** directed arcs

A spanning tree is a **connected network** without unconnected nodes

(e) Spanning tree: all nodes connected by directed arcs

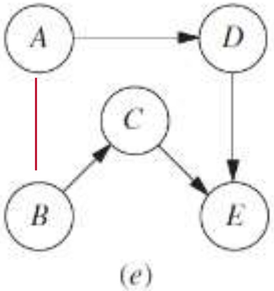


A spanning tree connects *n* nodes with *n-1* directed arcs

A spanning tree is a **connected network** without unconnected arcs

n-1 is both the **minimum** number of arcs needed and the **maximum** one, as adding one arc would generate an undirected **cycle**

Adding e.g. arc A→C (even if undirected) closes the loop but generates undirected cycles



We are now ready to tackle the **shortest path problem**

Ramon Casas and Pere Romeu on a Tandem, Barcelona. Source: Wikipedia Commons

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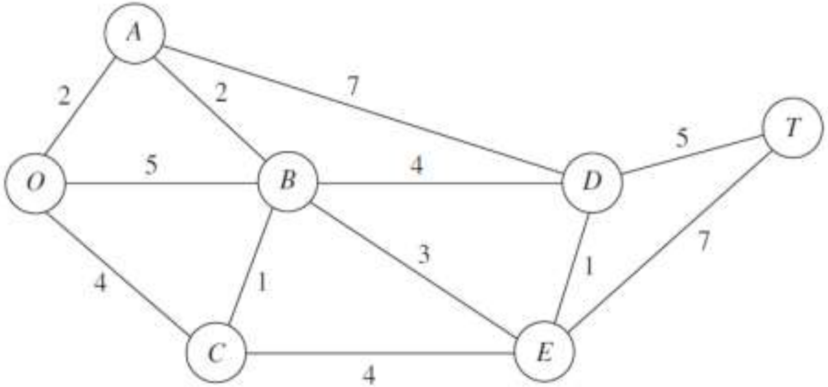
“Consider an undirected and connected network with two special nodes called the origin and the destination. Associated with each of the links (**undirected arcs**) is a nonnegative distance. The objective is to find the shortest path (the path with the minimum total distance) from the origin to the destination”



Let’s learn by doing, on our test case: the mission is to go from the entrance O to the scenic point T

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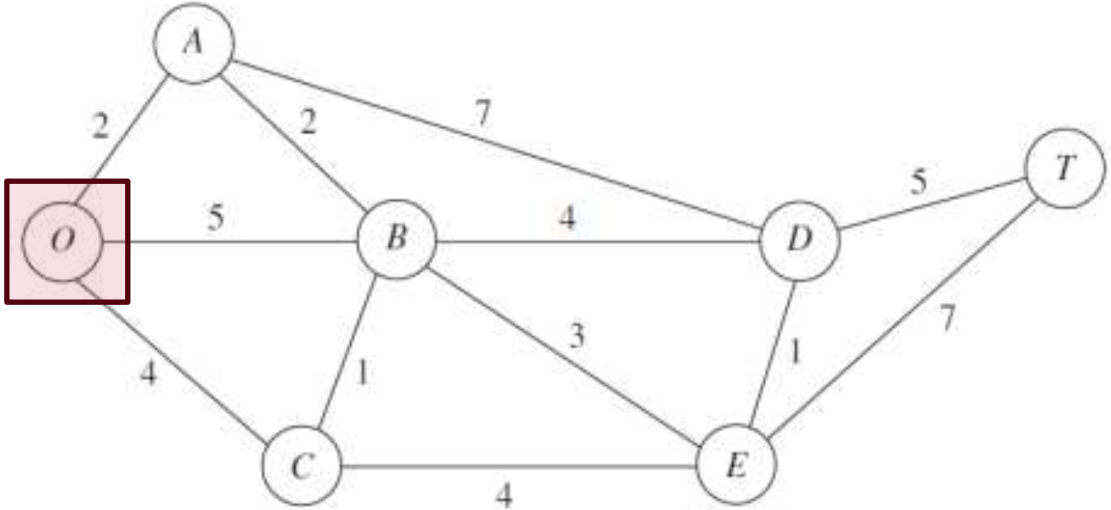
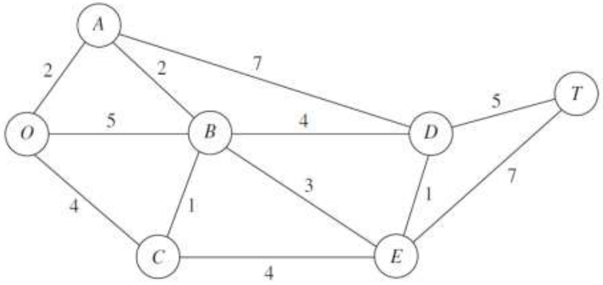
Algorithm for the Shortest-Path Problem



Theory: Objective of nth iteration: Find the nth nearest node to the origin (to be repeated for $n = 1, 2, ...$ until the nth nearest node is the destination.
Practice: the nearest note to O is A



Theory: Objective of nth iteration: Find the nth nearest node to the origin (to be repeated for $n = 1, 2, ...$ until the nth nearest node is the destination.
Practice: the nearest note to O is A

TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*

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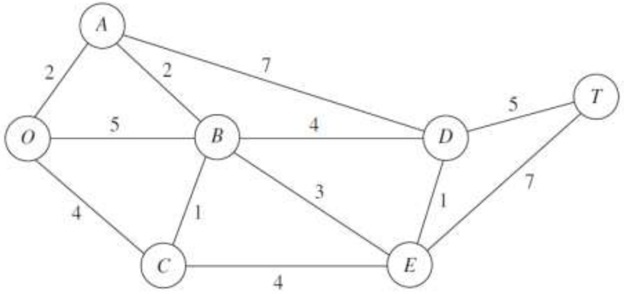
Algorithm for the Shortest-Path Problem



Theory: Input needed for nth iteration: $n-1$ nearest nodes to the origin (solved for at the previous iterations), including their shortest path and distance from the origin. (These nodes, plus the origin, will be called solved nodes; the others are unsolved nodes)

Theory: Candidates for nth nearest node: Each solved node that is directly connected by a link to one or more unsolved nodes provides one candidate — the unsolved node with the shortest connecting link to its solved node is taken

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Theory: Candidates for nth nearest node: Each solved node (O, A now) that is directly connected by a link to one or more (nearest) unsolved nodes (C, B respectively) provides one candidate — the unsolved node with the shortest connecting link to this solved node. (Ties provide additional candidates)

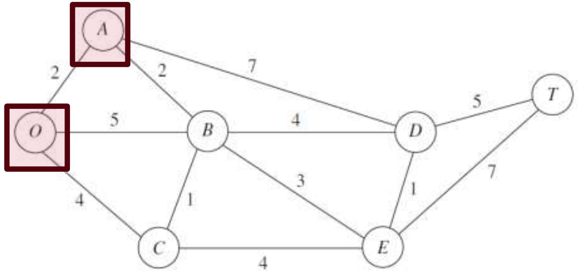

TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*

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Theory: <u>Calculation of nth nearest node</u>: For each such solved node and its candidate, add the distance between them and the distance of the shortest path from the origin to this solved node. The candidate with the smallest such total distance is the nth nearest node (ties provide additional solved nodes – as in this case C and B with 4 miles), and its shortest path is the one generating this distance

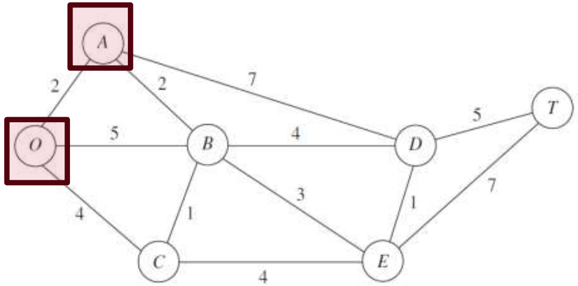

TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*

tie

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The solved nodes are now A, B, C, and the closest nodes are D, E

(E is closest for both B and C)

E wins as 4th closest node (7 miles)

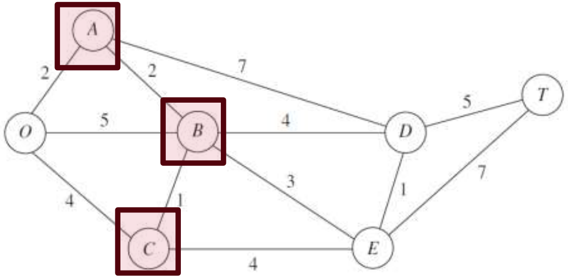

TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	nth Nearest Node	Minimum Distance	Last Connection
1	O	A	2	A	2	OA
2, 3	O A	C B	4 2 + 2 = 4	C B	4 4	OC AB
4	A B C	D E E	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	E	7	BE

wins

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The solved nodes closest to an unsolved note are now A, B, E, and for all the closest node is D
D wins as 5th closest node (8 miles)

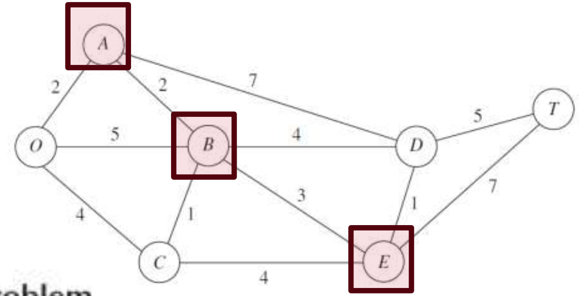

TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*
5	*A* *B* *E*	*D* *D* *D*	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	*D* *D*	8 8	*BD* *ED*

tie

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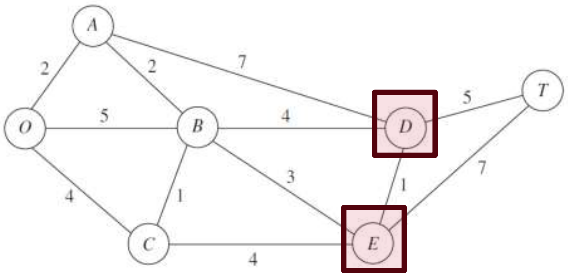
The solved nodes closest to an unsolved note are now D, E, and for both the closest node is the target destination T ; T wins as 6th closest node (13 miles)



■ **TABLE 10.2** Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*
5	*A* *B* *E*	*D* *D* *D*	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	*D* *D*	8 8	*BD* *ED*
6	*D* *E*	*T* *T*	8 + 5 = 13 7 + 7 = 14	*T*	13	*DT*

wins

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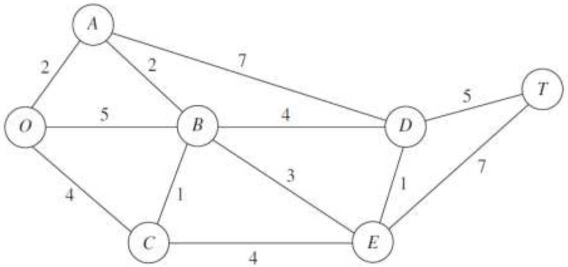
TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*
5	*A* *B* *E*	*D* *D* *D*	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	*D* *D*	8 8	*BD* *ED*
6	*D* *E*	*T* *T*	8 + 5 = 13 7 + 7 = 14	*T*	13	*DT*



Note how at each step the distance for the various candidate is computed…

… and the minimum distance is recorded

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TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

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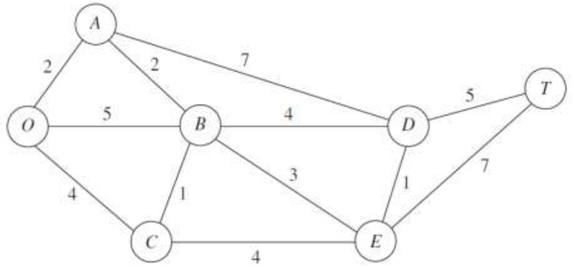
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*
5	*A* *B* *E*	*D* *D* *D*	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	*D* *D*	8 8	*BD* *ED*
6	*D* *E*	*T* *T*	8 + 5 = 13 7 + 7 = 14	*T*	13	*DT*

Focus on this column



We now move backword, from the destination to the origin

T→D→B→A→O

or

T→D→E→B→A→O

Both with 13 miles

Hence the solution:

O→A→B→D→T or

O→A→B→E→D→T

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■ **TABLE 10.2** Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
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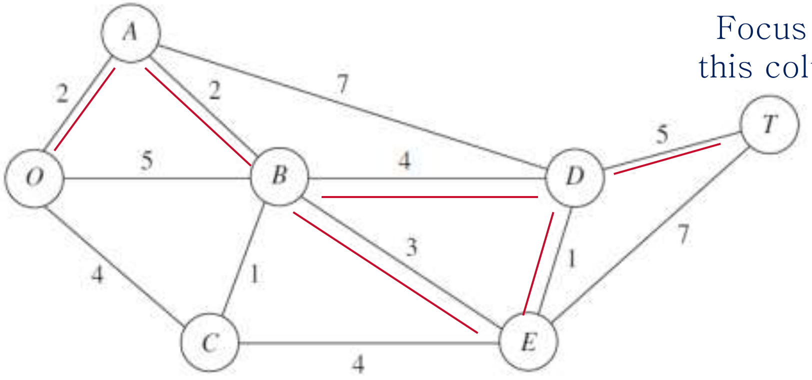
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*
5	*A* *B* *E*	*D* *D* *D*	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	*D* *D*	8 8	*BD* *ED*
6	*D* *E*	*T* *T*	8 + 5 = 13 7 + 7 = 14	*T*	13	*DT*

Focus on this column



T→D → B→A→O

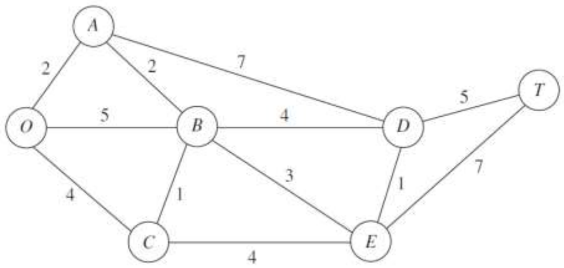
T→D → E→B→A→O

Perhaps clearer in this tree formulation?

Hence the solution:
O→A→B→E→D→T
or
O→A→B→D→T

Three practical problems

- **Shortest path** from entrance O to scenic point T
- Minimum length of telephone lines covering all tracks (**minimum spanning tree**)
- **Maximum flow** of mini-trains carrying non trekkers from entrance O to scenic point T



Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

Three practical problems

- **Shortest path** from entrance O to scenic point T ← Solved

Source: https://www.klook.com/en-US/activity/28218-yosemite-park-giant-sequoia-day-tour-san-francisco/?

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The Minimum Spanning Tree problem

Source: https://eu.palmbeachdailynews.com/story/entertainment/house-home/2019/12/15/palm-beach-gardening-help-save-planet-by-planting-these-native-trees/2079095007/

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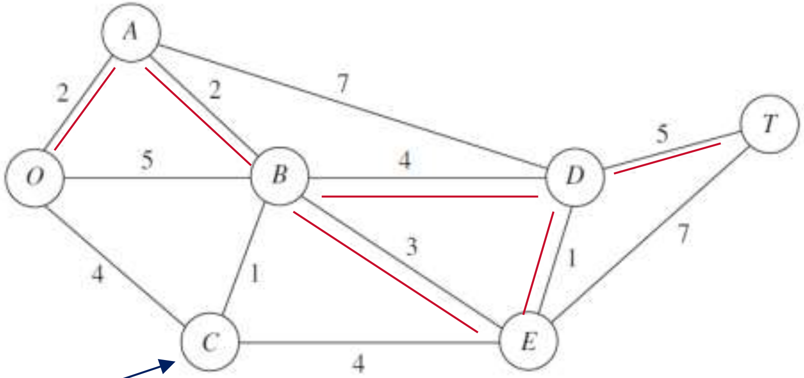
The Minimum Spanning Tree problem

For the shortest-path problem, we were looking for links that provide a path between the origin and the destination. We now just look for a minimum set of links that connect all nodes

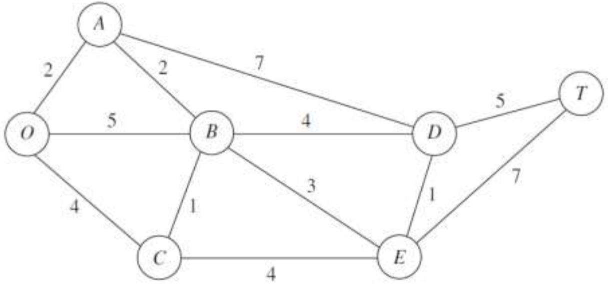
Could this be a spanning tree?



No, missing node

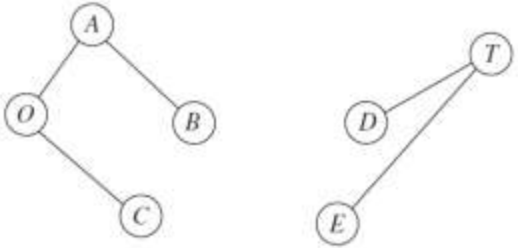
n nodes take *n-1* links

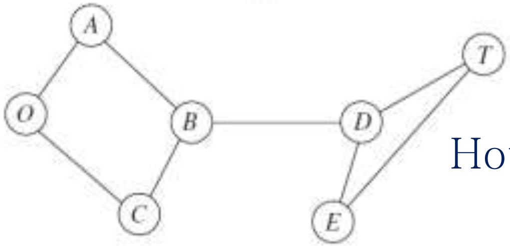
➔ Design the network by inserting enough links to satisfy the requirement that there be a path between every pair of nodes; The objective is to satisfy this requirement in a way that minimizes the total length of the links



Given these nodes

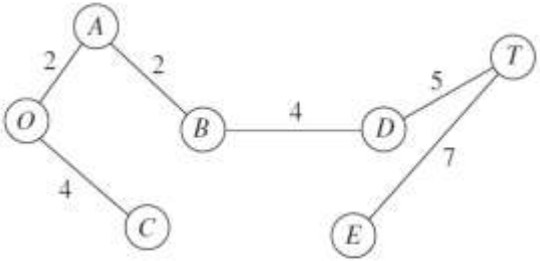
Could this be a spanning tree

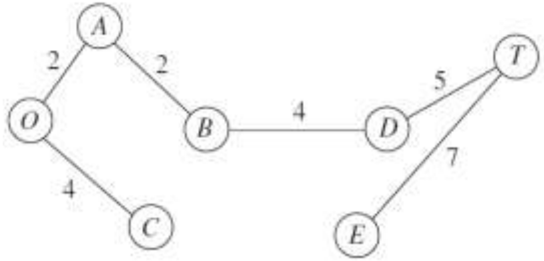




How about this?

Perhaps this?





All **7** nodes connected with **6** link

The strategy

Select arbitrarily a node

Identify closest unconnected node

Branch on ties (try both)

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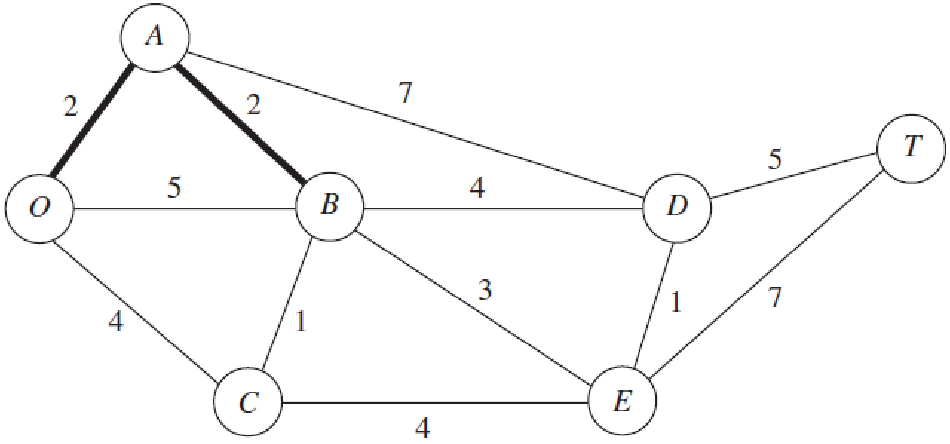
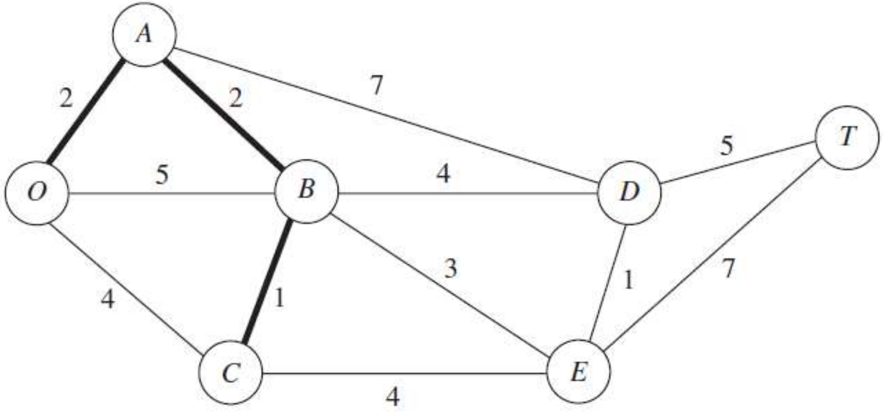
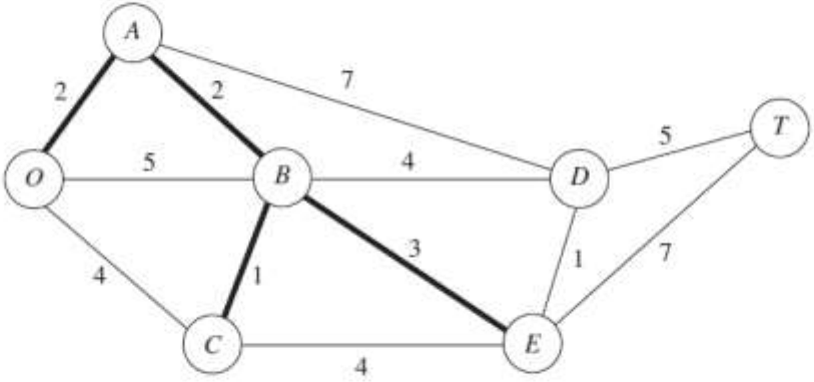


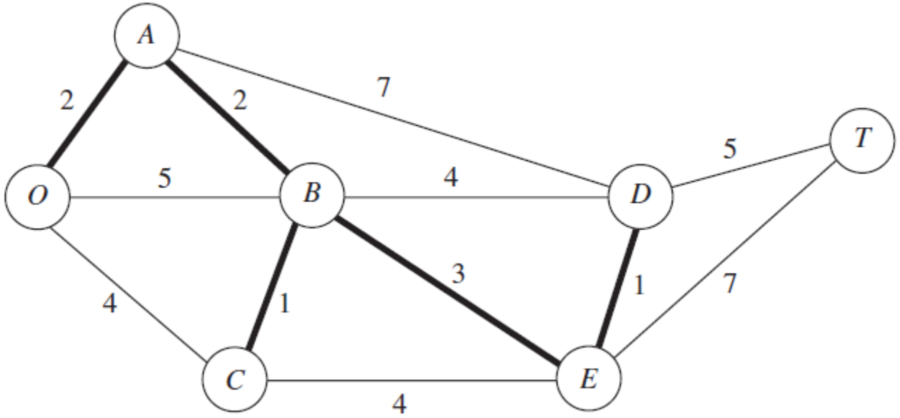
Select arbitrarily a node e.g. A
Identify closest unconnected node O or B
Branch on ties (try both)



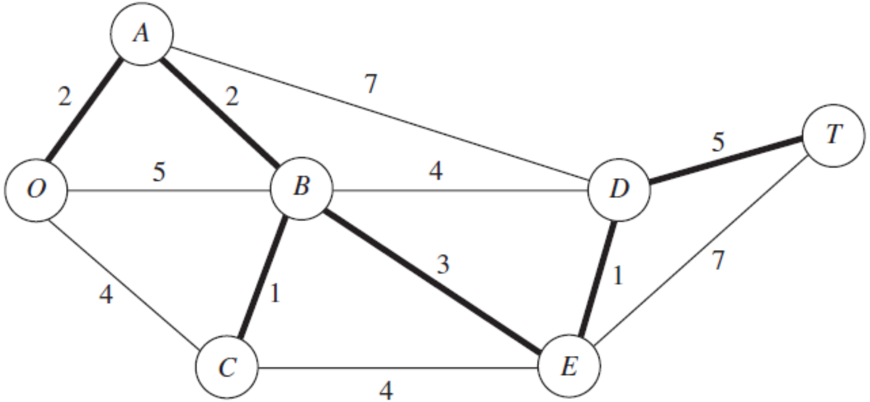
Identify closest unconnected node C



Identify closest unconnected node E



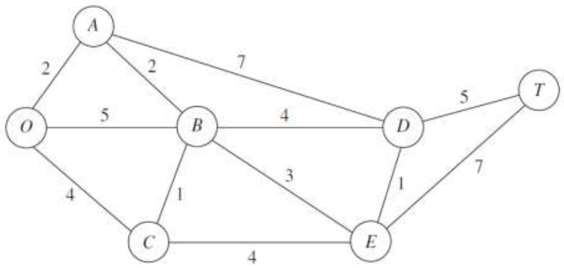
Identify closest unconnected node D



Here our spanning tree

Three practical problems

- **Shortest path** from entrance *O* to scenic point *T*
- Minimum length of telephone lines covering all tracks (**minimum spanning tree**)
- **Maximum flow** of mini-trains carrying non trekkers from entrance *O* to scenic point *T*



Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

- Minimum length of telephone lines covering all tracks (**minimum spanning tree**) ← Solved

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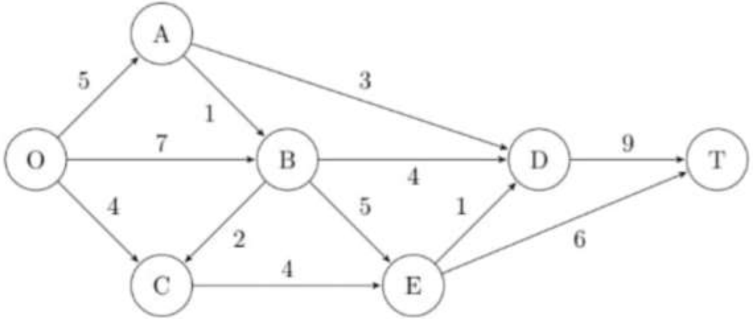
We are now left with the last problem to solve: **Maximum flow** of mini-trains carrying non trekkers from entrance O to scenic point T

Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/



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Maximum flow problem

"Typical kinds of applications of the maximum flow problem:

1. Maximize the flow through a company's distribution network from its factories to its customers.
2. Maximize the flow through a company's supply network from its vendors to its factories.
3. Maximize the flow of oil through a system of pipelines.
4. Maximize the flow of water through a system of aqueducts.
5. Maximize the flow of vehicles through a transportation network." (Hillier pp.387-388)

Source: https://www.livescience.com/61862-why-phantom-traffic-jams-happen.html

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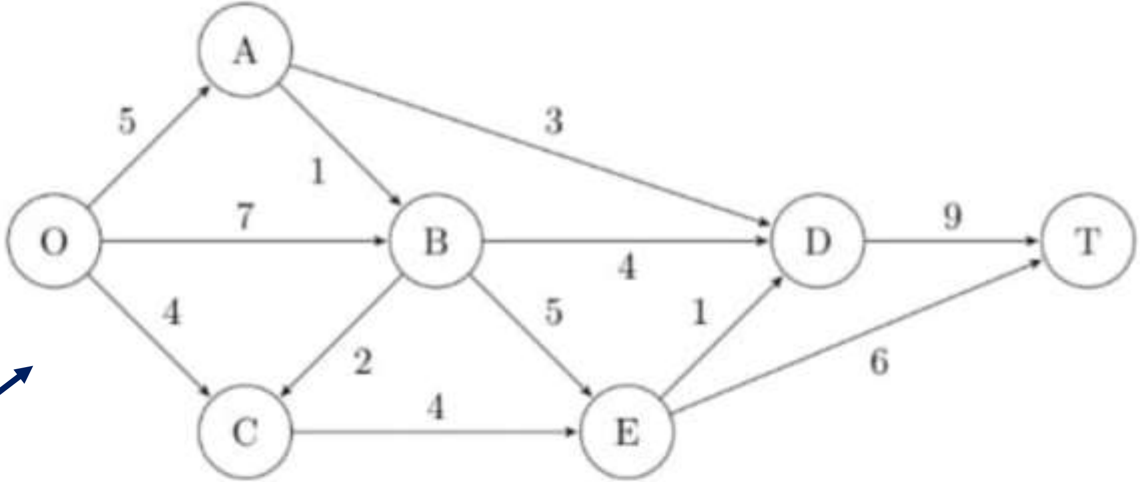
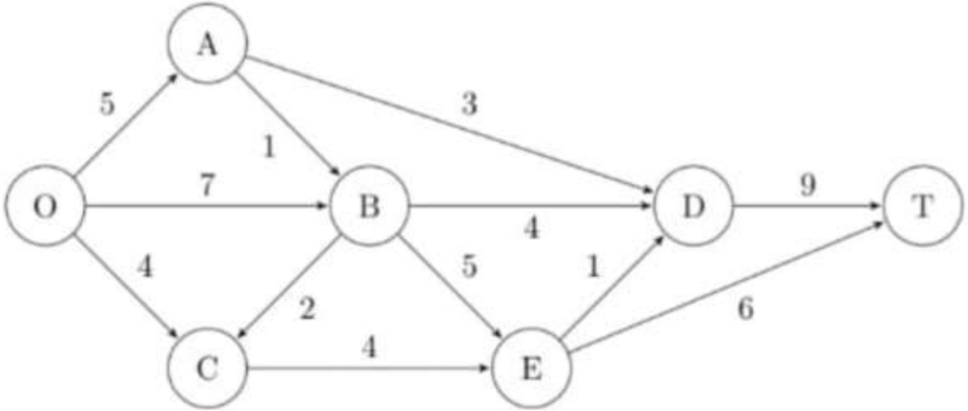
Maximum flow problem

Also here we proceed by a stepwise algorithm by ‘pumping’ items along preselected paths and recording changes. Numbers now represent maximum capacities. The path has now arrows. This is a directed network. **The mission is to move 14 trains**

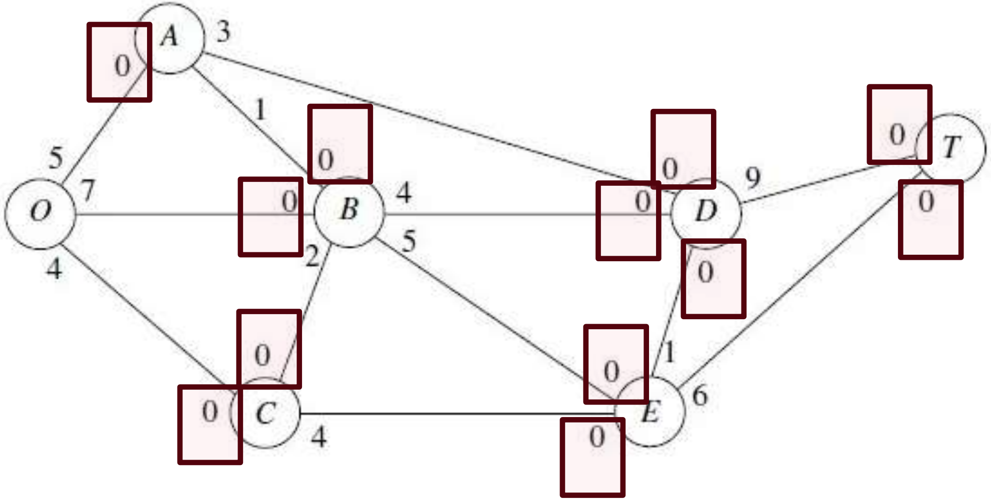
Warning: figures 10.6 and 10.7 in the online version https://www.andreasaltelli.eu/file/repository/Introduction_to_Operations_Research_10th_Frederick_S_Hillier.pdf are wrong, the others are right



This is right



Nothing has moved yet, and we note this by putting zeros **before** the node



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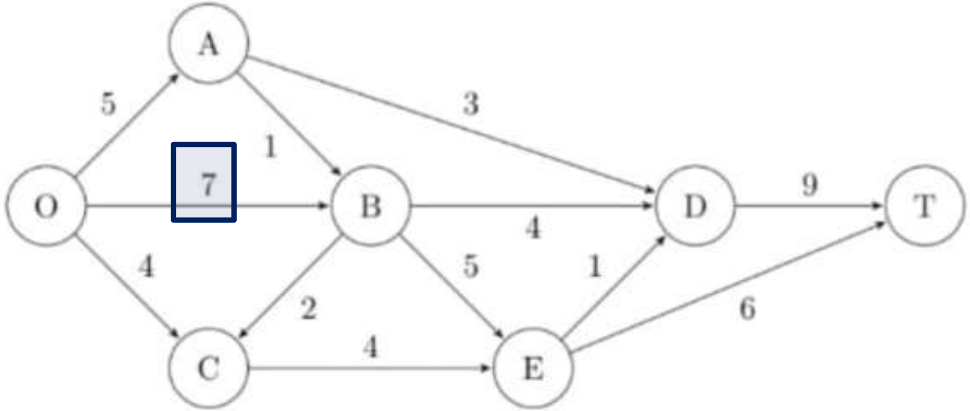
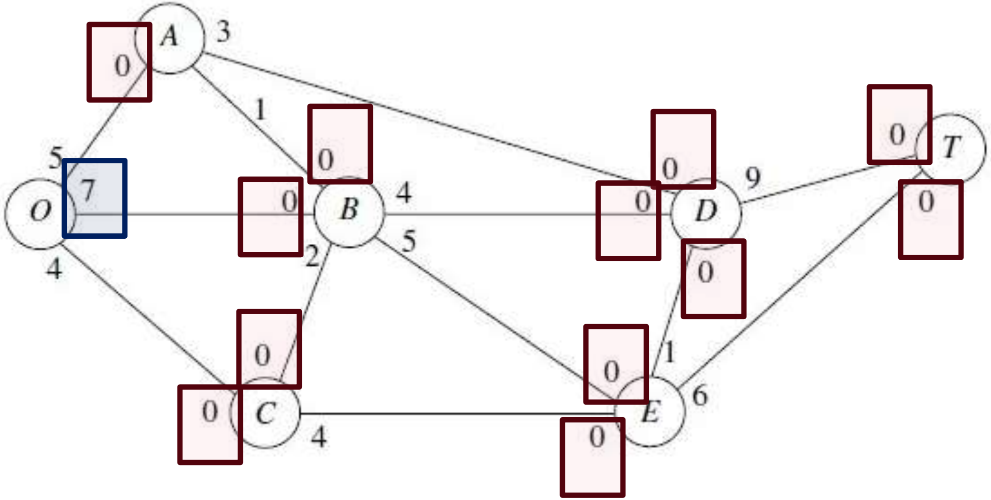
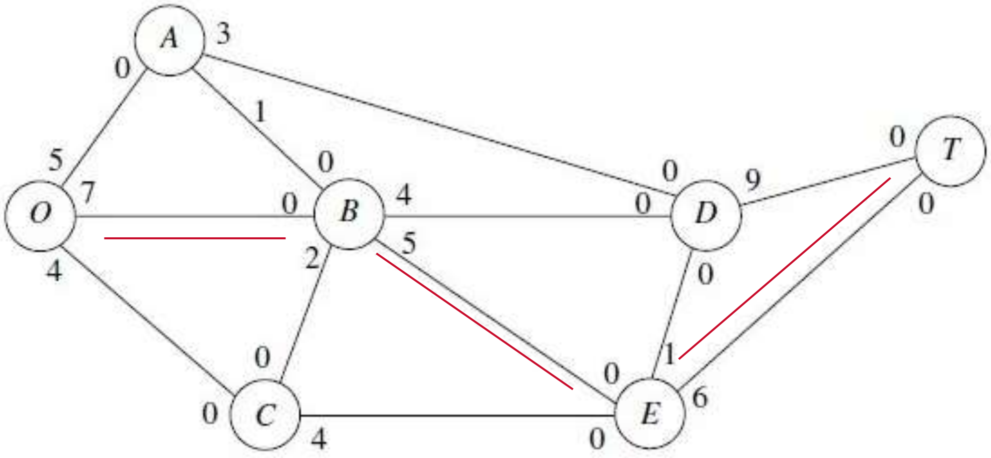
Note that in the last passage the capacity numbers (e.g. 7 for link OB) has been moved close to the node right **after** it



Interpret the zeroes **before** the node as 'nothing passed by here'





An augmenting path is a directed path from the source to the sink in the residual network such that every arc on this path has strictly positive residual capacity; for example

$$O \rightarrow B \rightarrow E \rightarrow T$$

is an augmenting path

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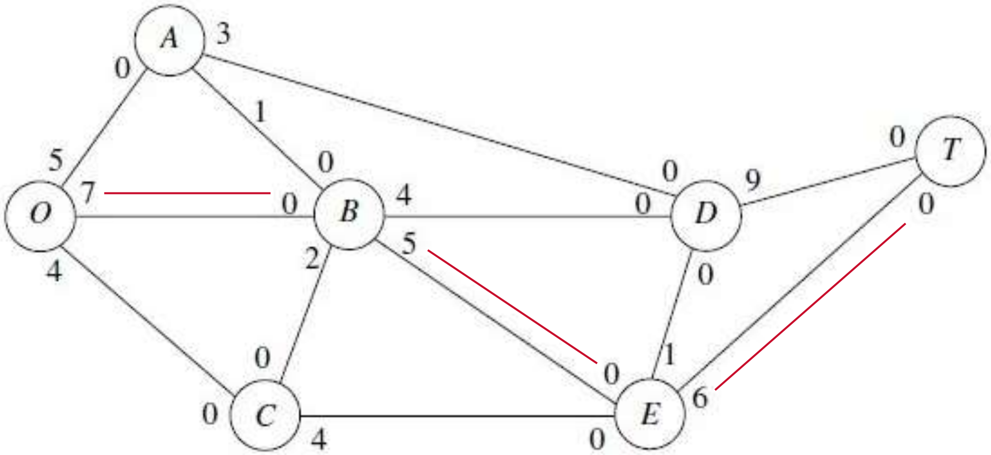
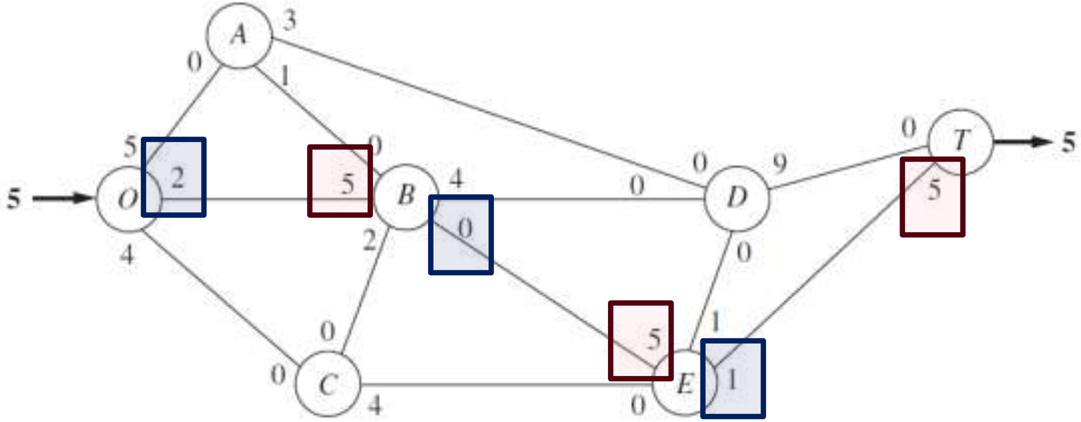
Chose now the smallest **residual capacity** on this path – among **7,5,6** → **5** is the smallest. Move five through this path, noting what happens



The capacity of link ***BE*** is now exhausted

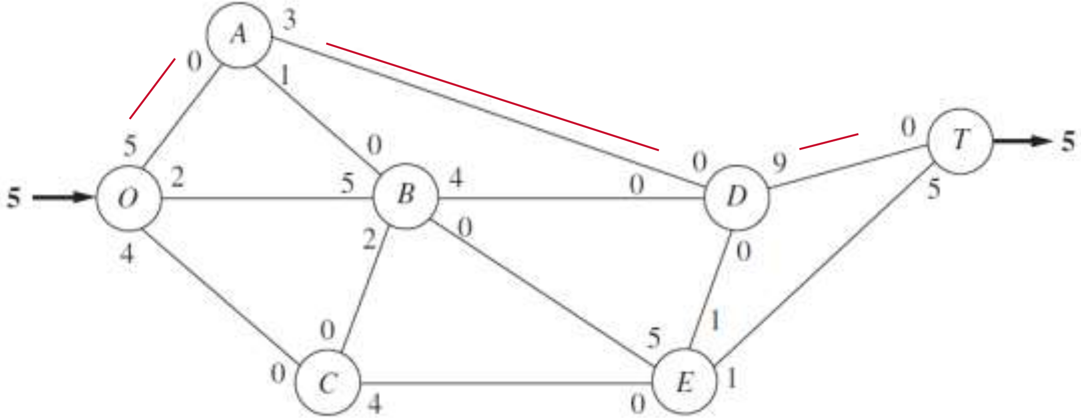


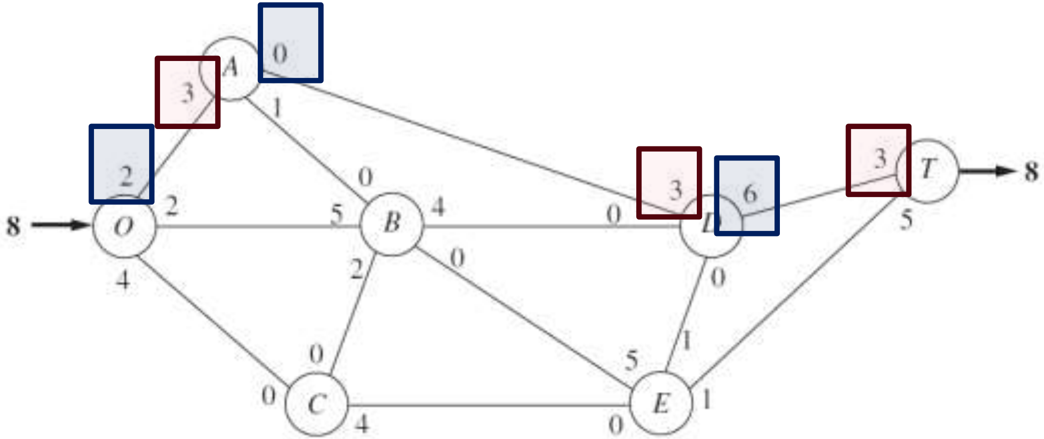
We now go to the augmenting path

$O \to A \to D \to T$

where the smallest capacity is 3, and move it

The capacity of link AD is now exhausted





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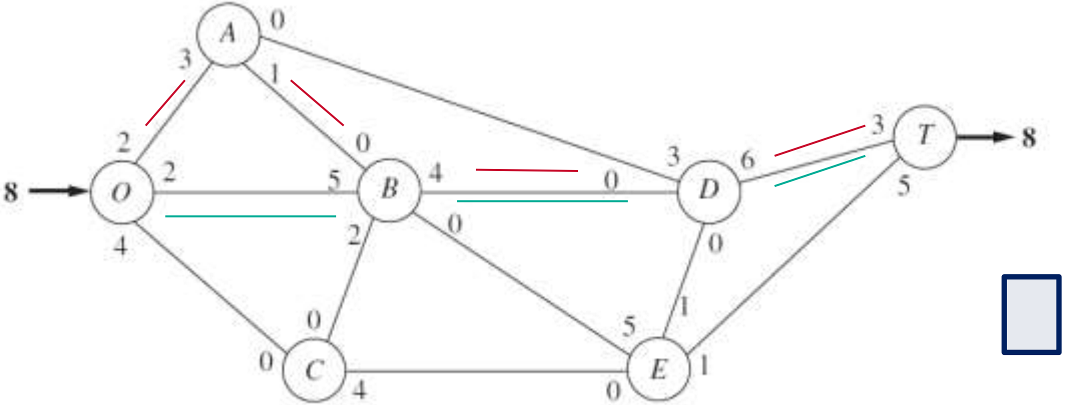
Assign a flow of 1 to the augmenting path

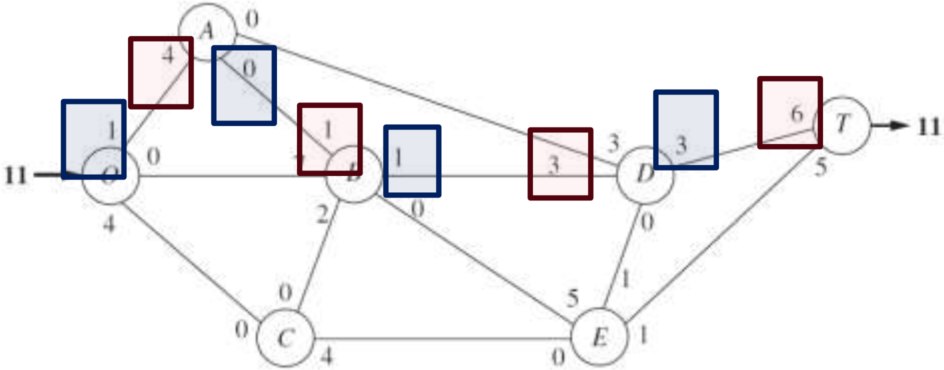
$O \to A \to B \to D \to T$

Assign a flow of 2 to the augmenting path

$O \to B \to D \to T$

The capacity of links AB and OB are now exhausted





Marking just one of the two paths

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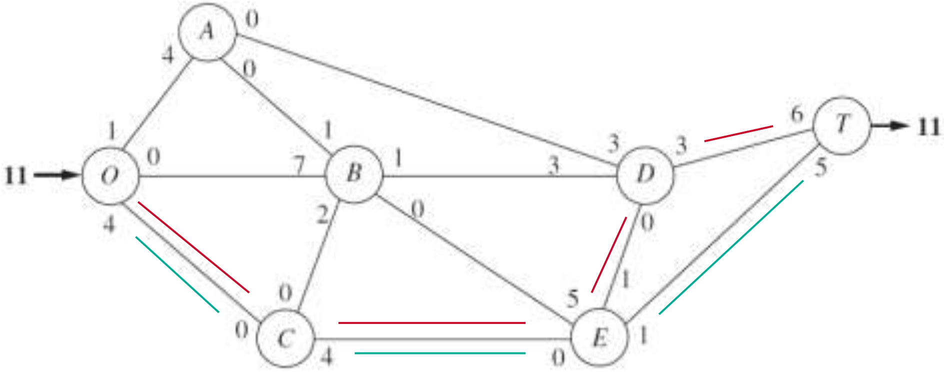
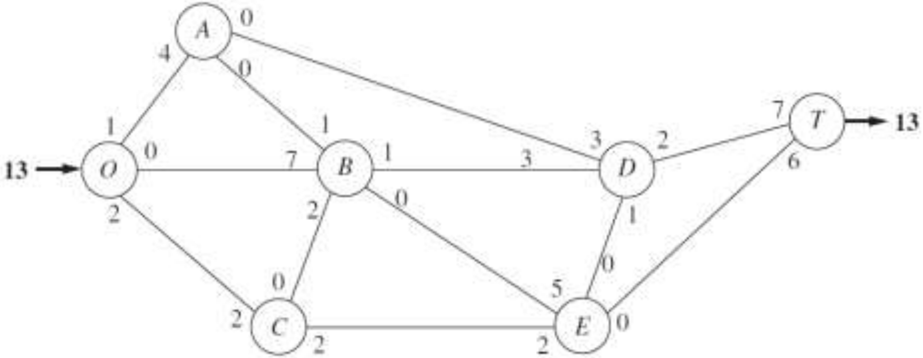
Assign a flow of 1 to the augmenting path

$O \rightarrow C \rightarrow E \rightarrow D \rightarrow T$



Assign a flow of 1 to the augmenting path

$O \rightarrow C \rightarrow E \rightarrow T$



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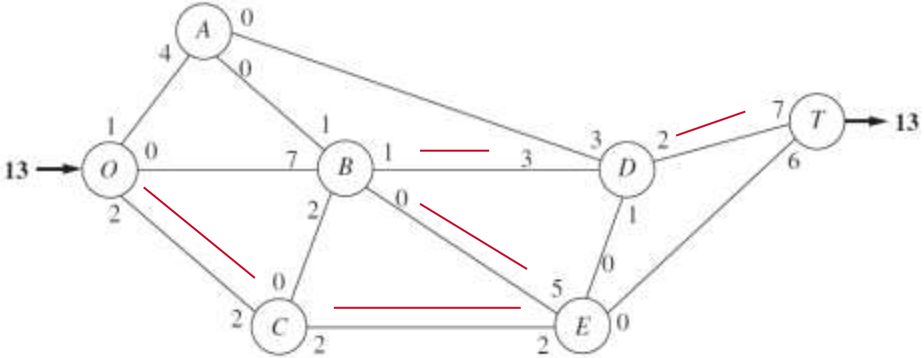
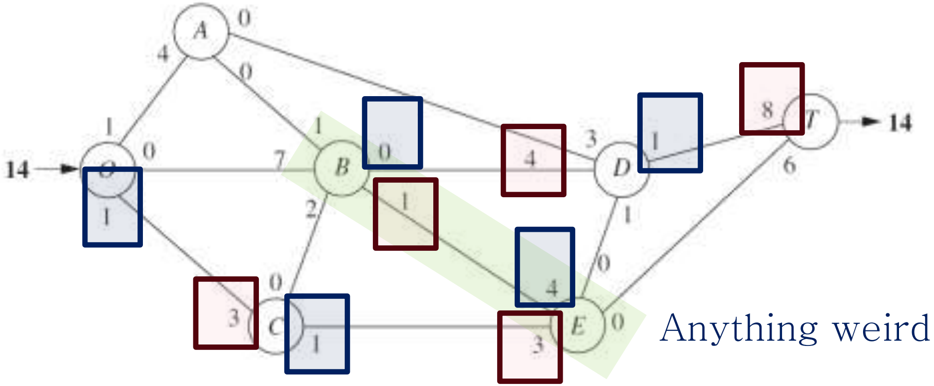
Assign a flow of 1 to the augmenting path

$O \rightarrow C \rightarrow E \rightarrow B \rightarrow D \rightarrow T$

The capacity of link BD is now exhausted





Anything weird here?

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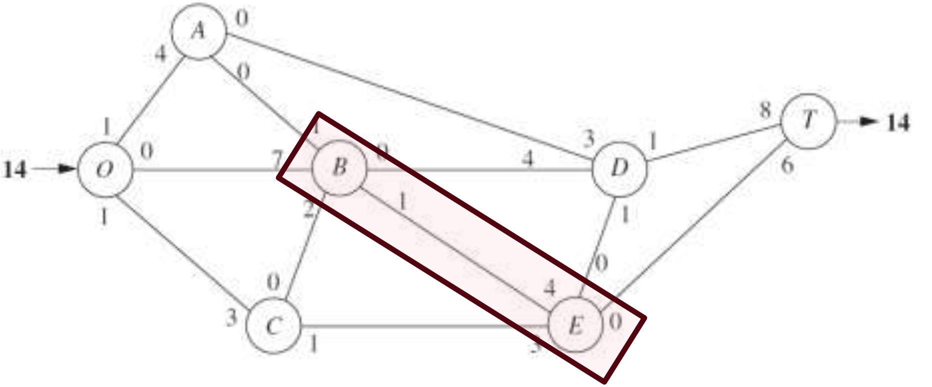
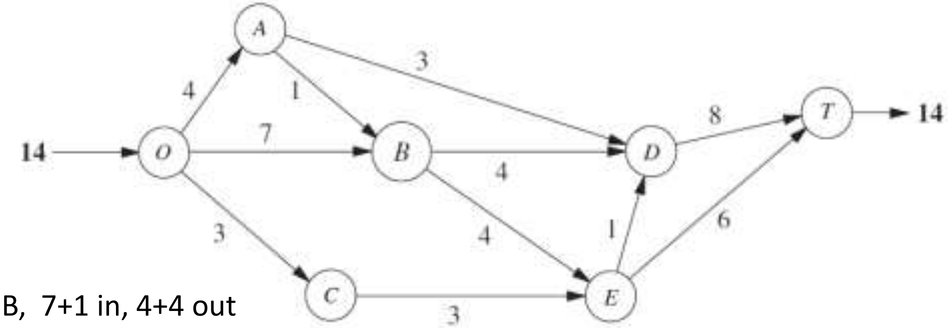
We have moved 'counter-current', but we haven't: **this this is the same as reversing part of a previous flow.** Check this out:



This was also the final move

Check node A, 4 in, 3+1 out



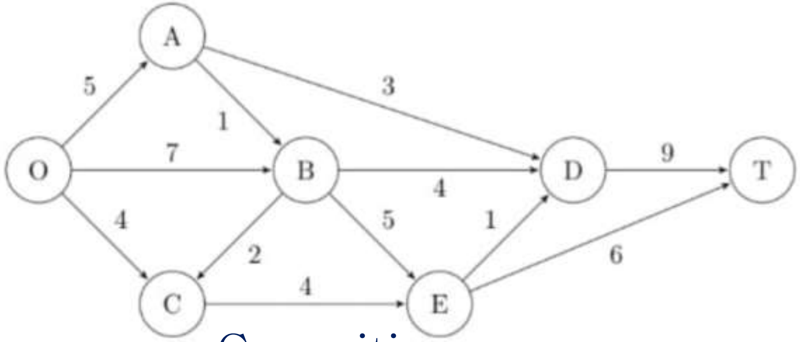
Check node B, 7+1 in, 4+4 out

Check node E, 7 in, 1+6 out

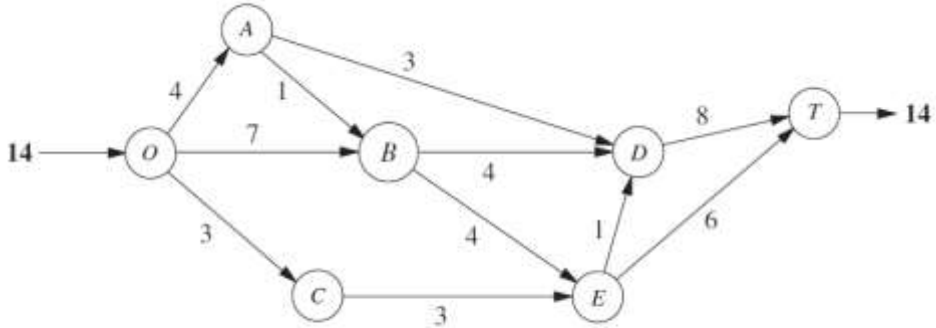
…etc

Check for yourself that

- No capacity has been violated
- No accumulation takes place at any node (what got in, got out)



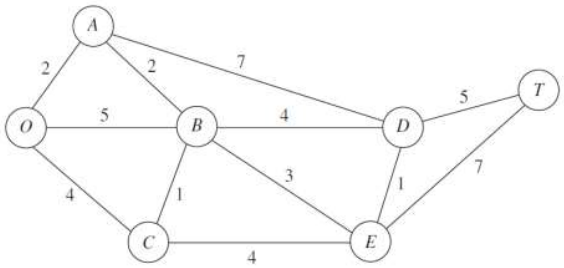
Capacities



Actual transport

Three practical problems

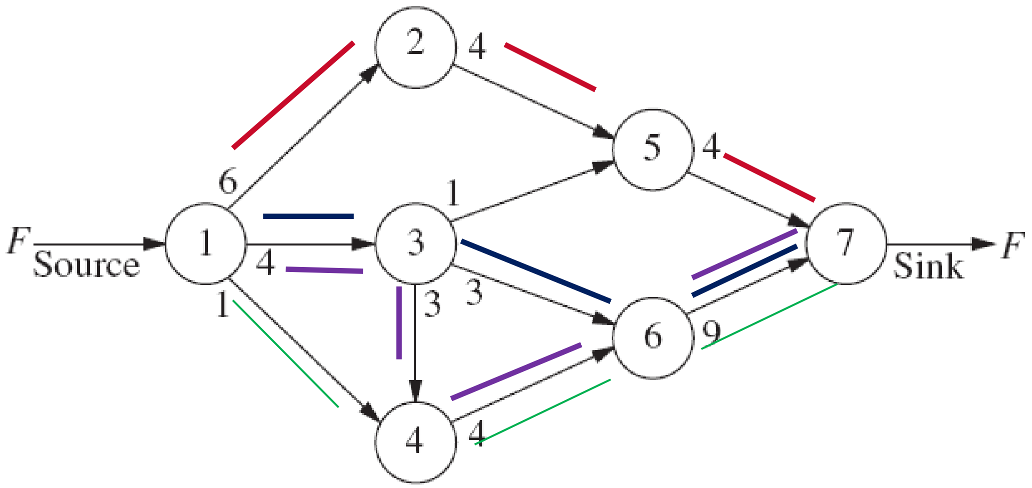
- **Shortest path** from entrance O to scenic point T
- Minimum length of telephone lines covering all tracks (**minimum spanning tree**)
- **Maximum flow** of mini-trains carrying non trekkers from entrance O to scenic point T



Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

Maximum flow of mini-trains carrying non trekkers from entrance O to scenic point T ← Solved

Solve the maximum flow problem for this network (Hillier 10.5-1)



10.5-1.

Arc	(1, 2)	(1, 3)	(1, 4)	(2, 5)	(3, 4)	(3, 5)	(3, 6)	(4, 6)	(5, 7)	(6, 7)
Flow	4	4	1	4	1	0	3	2	4	5

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15.

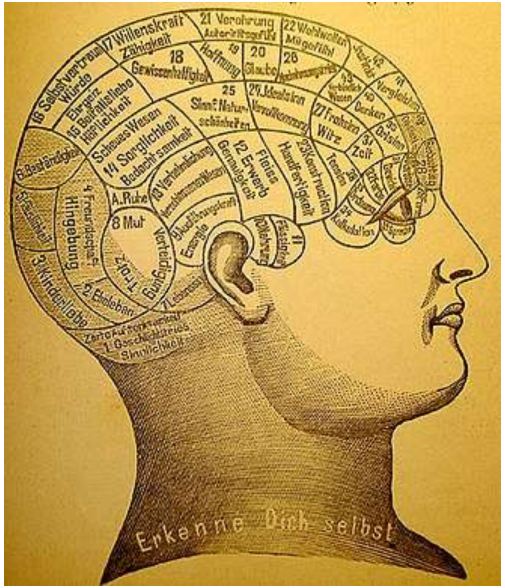
Integer Programming

Intuitions and fallacies. Why is it more difficult than LP. Integer and binary problems. Examples. Solution via branch and bound. Take home points. Hillier 2014, chapter 12.

Integer programming; intuition and fallacies

If the solutions need to be integer, there will be less of them, so Integer Programming (**IP**) will be easier than Linear Programming (**LP**)

- Yes, there will be less solutions, but still a very large numbers if they have to be found 'by inspection'
- The simplex solution of an IP treated as if it were an LP (what is called LP relaxation) generally generate unfeasible solutions



A phrenological mapping of the brain.
Source: Wikipedia Commons

Moving from LP to IP which of the four assumptions of LP will need to fall?

Proportionality: The contribution of each activity to the value of the objective function Z is proportional to the level of the activity x_j increase in Z that , as represented by the $c_j x_j$ term in the objective function

Additivity: Every function in a linear programming model (whether the objective function or the function on the left-hand side of a functional constraint) is the sum of the individual contributions of the respective activities

Divisibility: Decision variables in a linear programming model are allowed to have any values, including noninteger values, that satisfy the functional and nonnegativity constraints. Thus, these variables are not restricted to just integer values. Since each decision variable represents the level of some activity, it is being assumed that the activities can be run at fractional levels

Certainty: The value assigned to the parameters (the a_j^i's, b_i's, and c_j's) of a linear programming model are assumed to be known constants

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YES, NO decision variables

An important class of IP involves binary decision variables that can be represented as (0,1)

$$x_j = \begin{cases} 1 \text{ if decision} = \text{yes} \\ 0 \text{ if decision} = \text{no} \end{cases}$$

When this is the case the IP problem is said to be a Binary Integer Programming (BIP) problem

A prototype example: building or not building?

TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

Options	NPV
1fLA	9
1fSF	5
1fLA+w	NF
1fSF+w	5+4
2f	9+5
2f+wLA	NF
2f+wSF	NF

$$x_1 = \begin{cases} 1 \text{ if decision} = \text{yes build a factory in Los Angeles} \\ 0 \text{ if decision} = \text{no, don't build a factory in Los Angeles} \end{cases}$$

The choice is if building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building **at most one** new warehouse, but the choice of location is restricted to a city where a new factory is being built.

Just to be sure, which are the options?

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TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
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4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

$$x_1 = \begin{cases} 1 \text{ if decision} = \text{yes build a factory in Los Angeles} \\ 0 \text{ if decision} = \text{no, don't build a factory in Los Angeles} \end{cases}$$

The choice is if building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building at most one new warehouse, **but the choice of location is restricted to a city where a new factory is being built.**

➔ x_1 and x_2 can both be 1, but x_2 and x_3 will depend upon the choice made for x_1, x_2

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TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
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3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

Note: the capital required is already included in the net present value

$$x_1 = \begin{cases} 1 \text{ if decision} = \text{yes build a factory in Los Angeles} \\ 0 \text{ if decision} = \text{no, don't build a factory in Los Angeles} \end{cases}$$

It is easy to see that the function to be maximized is

$$Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

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TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
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4	Build warehouse in San Francisco?	x_4	$4 million	$2 million
			Capital available:	$10 million

$$x_1 = \begin{cases} 1 \text{ if decision} = \text{yes build a factory in Los Angeles} \\ 0 \text{ if decision} = \text{no, don't build a factory in Los Angeles} \end{cases}$$

And an evident constraint is

$$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

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TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
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4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

$$x_1 = \begin{cases} 1 \text{ if decision} = \text{yes build a factory in Los Angeles} \\ 0 \text{ if decision} = \text{no, don't build a factory in Los Angeles} \end{cases}$$

Note: $x_3 = \text{yes}$ only if $x_1 = \text{yes}$

Likewise: $x_4 = \text{yes}$ only if $x_2 = \text{yes}$

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TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
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3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

$x_3 = 1$ only if $x_1 = 1$
$x_4 = 1$ only if $x_2 = 1$

So, knowing that al variables need to be either 0 or 1 a possible way to include this contingency is the constraint

$$x_3 \leq x_1$$
$$x_4 \leq x_2$$

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TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
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4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

So, knowing that al variables need to be either 0 or 1 a possible way to include this contingency is the constraint

$$x_3 \leq x_1$$
$$x_4 \leq x_2$$

Since we only want at most one warehouse, it should also be
$x_3 + x_4 \leq 1$

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■ **TABLE 12.1** Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
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3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

Wrapping up, here the BIP problem:

$x_3 \leq x_1$

$x_4 \leq x_2$

rewritten in standard form →

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$

Subject to:

$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$

$-x_1 + x_3 \leq 0$

$-x_2 + x_4 \leq 0$

$x_3 + x_4 \leq 1$

and

$x_j \leq 1$

$x_j \geq 0$

x_j integer for $j = 1,2,3,4$

or

x_j binary for $j = 1,2,3,4$

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How many problems can be framed as BIP?

Investment decisions

Each yes-or-no decision:

Should we make a certain fixed investment?

Decision variable $x_j = \begin{cases} 1 & \text{if yes} \\ 0 & \text{if no} \end{cases}$

Siting decision

Each yes-or-no decision:

Should a certain site be selected to build a facility?

Decision variable $x_j = \begin{cases} 1 & \text{if yes} \\ 0 & \text{if no} \end{cases}$

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How many problems can be framed as BIP?

Relocating/restructuring, etc.?

- Each yes-or-no decision:
- Should a certain plant remain open?
- Should a certain site be selected for a new plant?
- Should a certain distribution center remain open?
- Should a certain site be selected for a new distribution center?

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How many problems can be framed as BIP?

Source: Wikipedia Commons

Dispatching decisions

Each yes-or-no decision:
Should a certain route be selected for one of the trucks?

Decision variable $x_j = \begin{cases} 1 & \text{if yes} \\ 0 & \text{if no} \end{cases}$

Or in more complicated arrangements: Should all the following be selected simultaneously for a delivery run:
1. A certain route,
2. A certain size of truck, and
3. A certain time period for the departure?

Decision variable $x_j = \begin{cases} 1 & \text{if yes} \\ 0 & \text{if no} \end{cases}$

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How many problems can be framed as BIP?

An airline application: Assigning crews to sequences of flights (crew scheduling problem). In a previous step of the analysis 12 crew flight sequences (ordered from one to a max of five), and the problem is to choose three of them so that all flights would be covered

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

Z is easy: If x_j =(0,1) decides if assigning the sequence to one of the three crews, then we must minimize:

$$Z = 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$$

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

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Since the crews are three it must be

$$\sum_{j=1}^{12} x_j = 3$$

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
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2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

Then for each of the 11 flights (1. San Francisco to Los Angeles all the way to 11. Seattle to Los Angeles) it must be that the sum of the coefficients covering that flight add up to one or more (more crews can fly on a flight – there can be a non working crew that still needs to be paid)

1. $x_1 + x_4 + x_7 + x_{10} \geq 1$
2. $x_2 + x_5 + x_8 + x_{11} \geq 1$

…

11. $x_6 + x_9 + x_{10} + x_{11} + x_{12} \geq 1$

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

Never mind the numbers 1-5 (flight sequence) in this region of the table – it is only important whether there is a number or not

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So wrapping up the problem is:

Minimize $Z = 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$

Subject to

$\sum_{j=1}^{12} x_j = 3$ and the 11 constraints

$x_1 + x_4 + x_7 + x_{10} \geq 1$
$x_2 + x_5 + x_8 + x_{11} \geq 1$
…
$x_6 + x_9 + x_{10} + x_{11} + x_{12} \geq 1$

Are we done?
x_j binary for $j = 1,2, \ldots 12$

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

Minimize

$Z = 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$

Verify that one optimal solution for this BIP model is

$x_3 = 1$ (assign sequence 3 to a crew)

$x_4 = 1$ (assign sequence 4 to a crew)

$x_{11} = 1$ (assign sequence 11 to a crew)

and all other x_j =0

and that another optimal solution is

$x_1 = 1$

$x_5 = 1$

$x_{12} = 1$

and all other x_j =0

And compute Z for the two options

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

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Minimize

$$Z = 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$$

Verify that one optimal solution for this BIP model is
$x_3 = 1$ (assign sequence 3 to a crew)
$x_4 = 1$ (assign sequence 4 to a crew)
$x_{11} = 1$ (assign sequence 11 to a crew)
and all other x_j =0

and that another optimal solution is

$x_1 = 1$
$x_5 = 1$
$x_{12} = 1$
and all other x_j =0

And compute Z for the two options

Z=18

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

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We just solved a **set covering problem**, (all flights need to be covered)

A related BIP is the **set partitioning problem**, where instead of e.g.

$$x_1 + x_4 + x_7 + x_{10} \geq 1$$

(previous problem) one would ask:

$$x_1 + x_4 + x_7 + x_{10} = 1$$

This would prevent more than one crew flying on the same flight

Source: https://airportwingspvtltd.wordpress.com/2016/01/04/role-and-responsibilities-of-cabin-crew/

As mentioned, IP are in general more difficult than LP; though there are less solutions, there are many of them; e.g. for a BIP with ten decision variables the number of possible solutions is $2^{10} = 1{,}024$

Why?
permutations with repetition of two elements in groups of 10

It is not forbidden to try a LP approach for a IP problem (**LP relaxation**), though in general there is no guarantee that the solution will be feasible for the IP

It is not forbidden to try a LP approach for a IP problem (**LP relaxation**), though in general there is no guarantee that the solution will be feasible for the IP

… but when the LP relaxation solution satisfies the integer restriction of the IP problem, this solution must be optimal for the IP problem as well (=the best among all LP solutions is also the best for the subset of the IP solutions)

The LP relaxation value for the optimization function Z is in any case an upper bound for the Z of the integer problem

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It is not forbidden to try a LP approach for a IP problem (**LP relaxation**), though in general there is no guarantee that the solution will be feasible for the IP

“Therefore, it is common for an IP algorithm to begin by applying the simplex method to the LP relaxation to check whether this fortuitous outcome has occurred”

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Exercise: (Hillier 12.1-2) A young couple, Eve and Steven, want to divide their main household chores (marketing, cooking, dishwashing, and laundering) between them so that each has two tasks but the total time they spend on household duties is kept to a minimum. Their efficiencies on these tasks differ, where the time each would need to perform the task is given by the following table:

	Time Needed per Week			
	Marketing	**Cooking**	**Dishwashing**	**Laundry**
Eve	4.5 hours	7.8 hours	3.6 hours	2.9 hours
Steven	4.9 hours	7.2 hours	4.3 hours	3.1 hours

- Write this as a binary integer programming problem
- Guess a solution

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More tricks with binary variables. From Hillier, example pages 489–491

When one of two constraints must hold, for example

$$3x_1 + 5x_2 - 7x_3 \leq 12$$

or

$$4x_1 + 2x_2 + x_3 \leq 15$$

But not both we can use an auxiliary binary variable y and impose

$$3x_1 + 5x_2 - 7x_3 \leq 12 + My$$
$$4x_1 + 2x_2 + x_3 \leq 15 + M(1 - y)$$
$$x_i \geq 0$$
$$y \text{ binary}$$

Where M is the usual large number.
If $y = 0$ the first constraint holds, if $y = 1$ the second

"It is common for an IP algorithm to begin by applying the simplex method to the LP relaxation to check whether this fortuitous outcome has occurred"

This may or may not work see e.g. the simple example

Maximize $Z = x_2$ subject to

$$-x_1 + x_2 \leq \frac{1}{2}$$
$$x_1 + x_2 \leq \frac{7}{2}$$

← Find graphically the <u>linear</u> solution of this problem

and

$$x_1 \geq 0, x_2 \geq 0$$
x_1, x_2 integers ← I.e. removing this constraint

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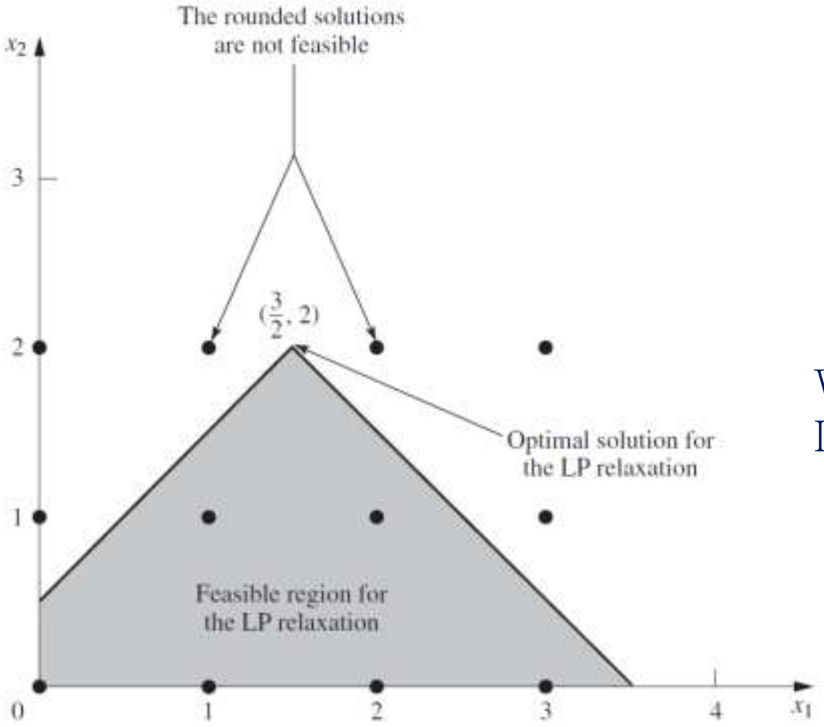

■ **FIGURE 12.2**
An example of an IP problem where the optimal solution for the LP relaxation cannot be rounded in any way that retains feasibility.

Which is instead the IP solutions?

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Another case where the relaxation solution can be **not OK**

Maximize $Z = x_1 + 5x_2$ subject to

$x_1 + 10x_2 \leq 20$

$x_1 \leq 2$

and

$x_1 \geq 0, x_2 \geq 0$

x_1, x_2 integers

Find graphically the <u>linear</u> solution of this problem ⋯

⋯ i.e. removing this constraint

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$$x_1 = 0,\quad x_2 = 2,\quad Z = 10$$

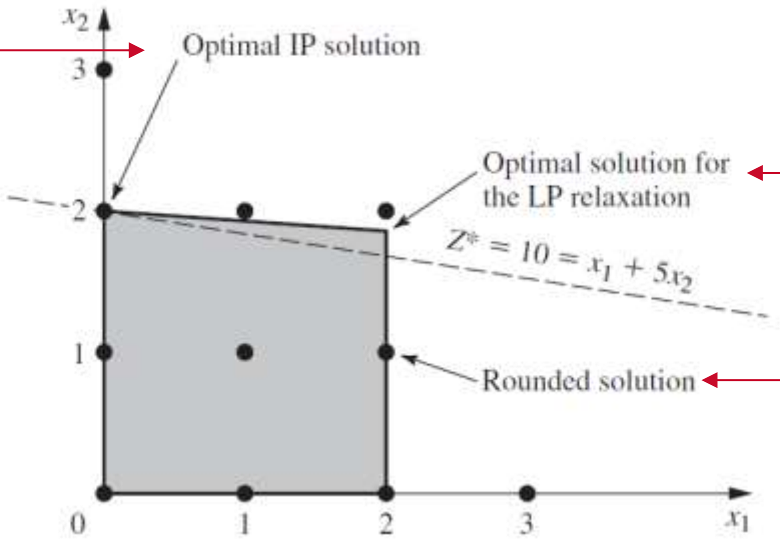

FIGURE 12.3
An example where rounding the optimal solution for the LP relaxation is far from optimal for the IP problem.

$$x_1 = 2,\quad x_2 = \frac{9}{5},\quad Z = 11$$

$$x_1 = 2,\quad x_2 = 1,\quad Z = 7$$

Did we violate the rule that the LP solution is an upper bound for the IP solution?

$$x_1 = 2,\quad x_2 = \frac{9}{5},\quad Z = 11$$

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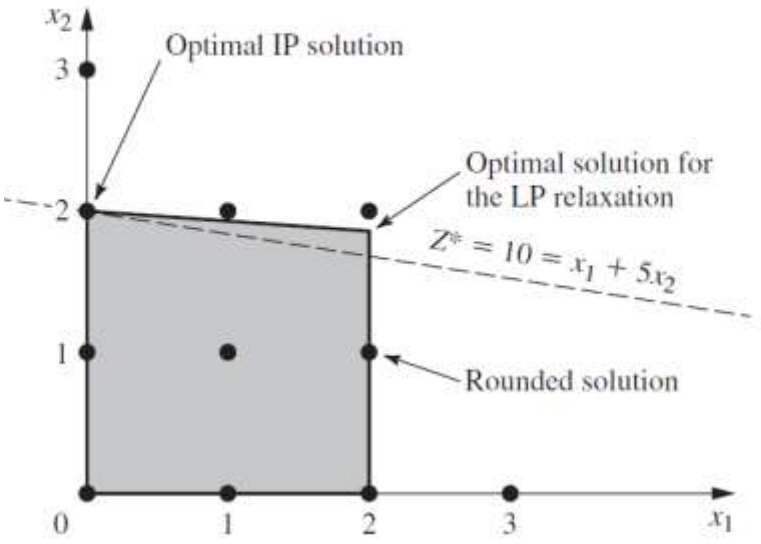
When there are may dimensions checking that the relaxation solution is OK can be tricky;

Here we have only 7 integer points in the feasible region, but the number of points grows exponentially with the number of dimensions



In many dimensions better use heuristic method (such as genetic algorithms, more later) that also work for nonlinear problems.

But there are IP problems whose structure guarantees an integer solution; remember the Transportation Problem (Section 12);

The integer solutions property: For transportation problems where every supply s_i and demand d_i have an integer value, all basic feasible (BF) solutions (including an optimal one) also have integer values

TABLE 9.3 Constraint coefficients for P & T Co.

	Coefficient of:												
	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	
A =	1	1	1	1									Cannery constraints
					1	1	1	1					
									1	1	1	1	
	1				1				1				Warehouse constraints
		1				1				1			
			1				1				1		
				1				1				1	

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But there are IP problems whose structure guarantees an integer solution; remember from the section on Transportation Problem (Section 12);

Source: Wikipedia Commons

Other special cases are the assignment problem, the shortest-path problem, and the maximum flow problem

Charles Chaplin's Modern Times, source http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

Ramon Casas and Pere Romeu on a Tandem, Barcelona. Source: Wikipedia Commons

Level of difficulty of LP versus IP

	Difficulty of LP problem	Difficulty of IP problem
Source	Number of constraints	Number of integer variables
		Binary or general integer?
		Special form?

Source:
https://www.dreamstime.com/illustration/accountant.html

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Back to out prototype example: building or not building?

TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

The choice is if building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building **at most one** new warehouse, but the choice of location is restricted to a city where a new factory is being built.

TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$

Subject to:

$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$

$-x_1 + x_3 \leq 0$

$-x_2 + x_4 \leq 0$

$x_3 + x_4 \leq 1$

and

x_j binary for $j = 1,2,3,4$

If we apply LP relaxation replacing

x_j binary for $j = 1,2,3,4$

with

$x_j \geq 0$ for $j = 1,2,3,4$

We obtain $x_1, x_2, x_3, x_4 = \left(\frac{5}{6}, 1,0,1\right)$

with $Z = 16.5$

We round this to 16 and keep it as an upper bound for the IP problem

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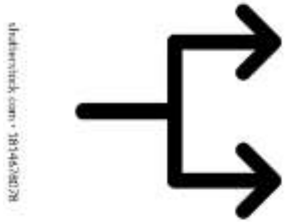
One method to solve IP problems: the branch-and-bound technique

- Branching (split the problem in two branches)
- Bounding (seek for a local optima for Z)
- Fathoming (Resolving the branching at fathomed the node)



Source: https://www.123rf.com/

Source: https://thesaurus.plus/synonyms/fathomed

- Branching (split the problem in two branches)

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

$x_1 = 0$

Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \leq 10$
$x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

$x_1 = 1$

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

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- Branching (split the problem in two branches)

We are splitting following the order of the variables, i.e. here starting by x_1. Better strategies are available

- Branching (split the problem in two branches)

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and
x_j binary for $j = 1,2,3,4$

$x_1 = 0$

Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \leq 10$
$x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and
$x_j \geq 0$ for $j = 2,3,4$

$x_1 = 1$

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and
$x_j \geq 0$ for $j = 2,3,4$

The two subproblems are treated as linear instead of integer

- Bounding (seek for local optima for Z)

- Branching (split the problem in two branches)

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10$
$-x_1 + x_3 \le 0$
$-x_2 + x_4 \le 0$
$x_3 + x_4 \le 1$
and
x_j binary for $j = 1,2,3,4$

$x_1 = 0$:

Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \le 10$
$x_3 \le 0$
$-x_2 + x_4 \le 0$
$x_3 + x_4 \le 1$
and
$x_j \ge 0$ for $j = 2,3,4$

$x_1 = 1$:

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \le 10$
$-1 + x_3 \le 0$
$-x_2 + x_4 \le 0$
$x_3 + x_4 \le 1$
and
$x_j \ge 0$ for $j = 2,3,4$

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Linear programming applied to these solutions yields

$x_1, x_2, x_3, x_4 = (0,1,0,1)$ with $Z = 9$

$x_1, x_2, x_3, x_4 = \left(1, \frac{4}{5}, 0, \frac{4}{5}\right)$ with $Z = 16.5$

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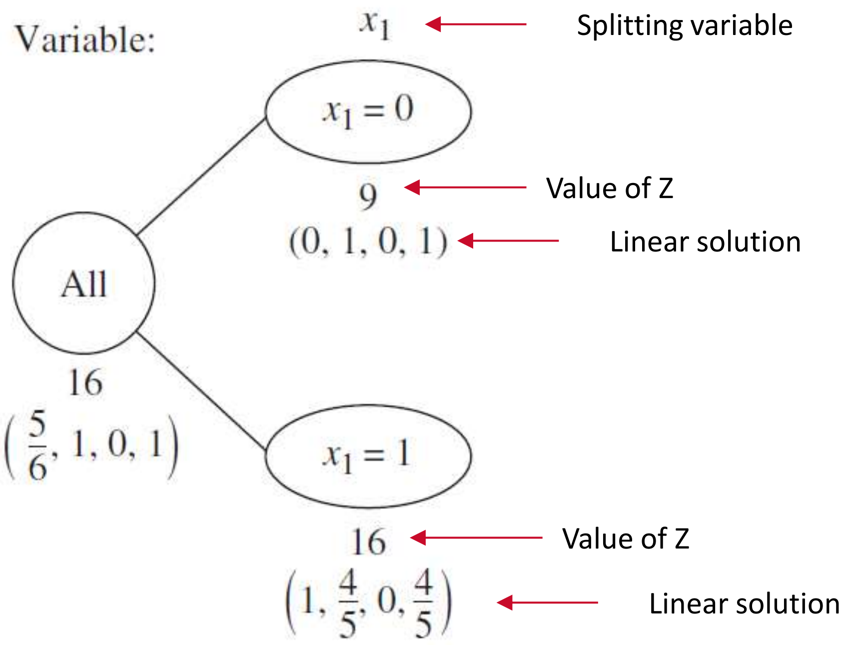
This is where we are at the end of the first bounding step:



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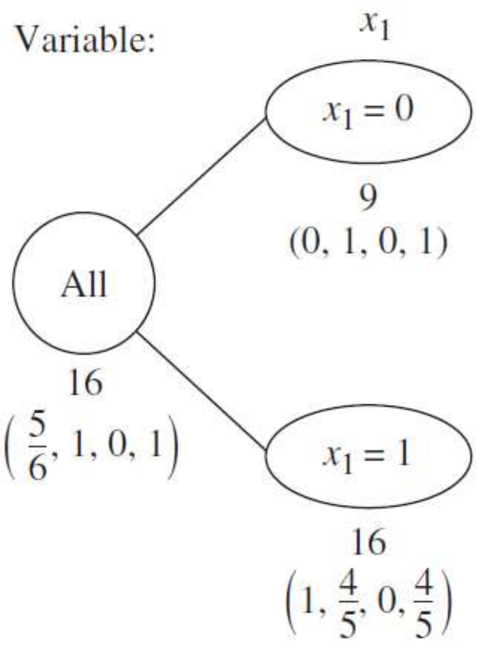
- Fathoming (Resolving the branching at fathomed the node)

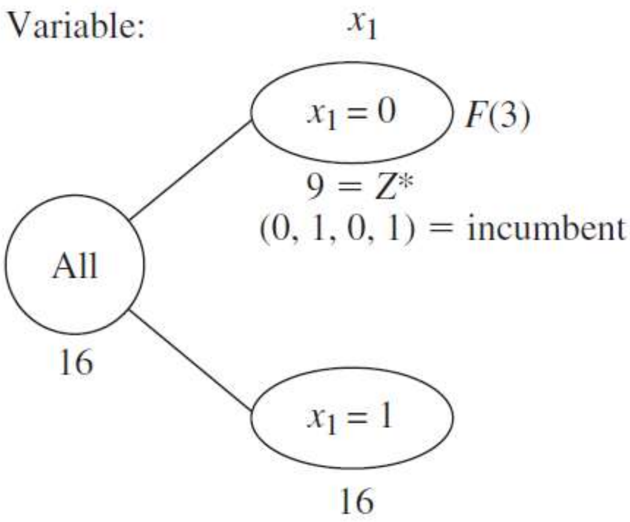


Since it is optimal it does not pay to search for other solutions in this branch

This solution is made of integers! It is **hence optimal for the subproblem with** $\boldsymbol{x_1 = 0}$. We call this now **the incumbent optimum** $Z^* = 9$ and say that the branch $x_1 = 0$ is fathomed; in the following we can get rid of all branches whose $Z \leq Z^* = 9$

This cannot be fathomed

- Fathoming (Resolving the branching at fathomed the node)



In fact, there are 3 ways of fathoming:

Test 1: Its bound by being $\leq Z^*$

Test 2: Its LP relaxation has no feasible solutions

Test 3: The optimal solution for its LP relaxation is integer.

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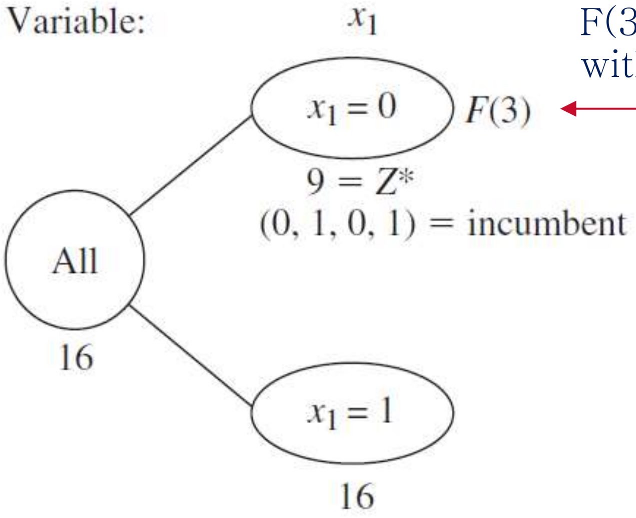
- Fathoming (Resolving the branching at fathomed the node)



F(3)=fathomed with test 3

In fact, there are 3 ways of fathoming:

Test 1: Its bound $\leq Z^*$

Test 2: Its LP relaxation has no feasible solutions

Test 3: The optimal solution for its LP relaxation is integer

If this solution is better than the incumbent, it becomes the new incumbent Z^*, and test 1 is reapplied to all previous unfathomed subproblems using this new larger Z^*

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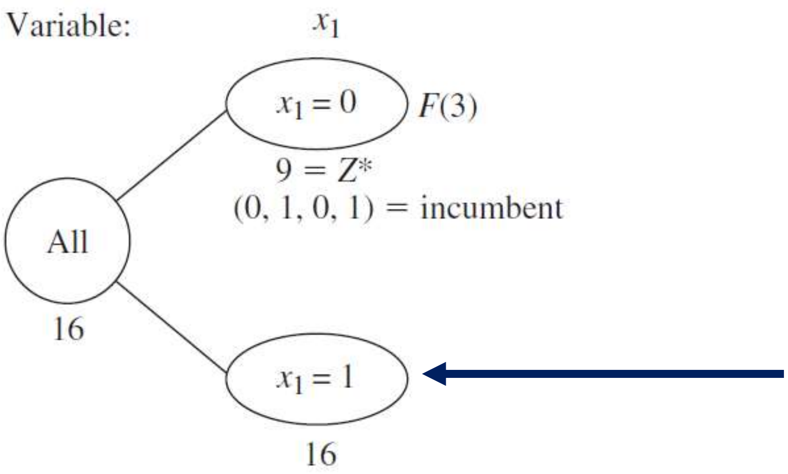
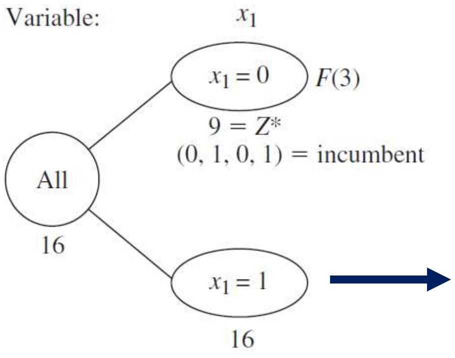
- Continuing the example



We now branch the $x_1 = 1$ problem by branching x_2 between 0 and 1

- Continuing the example



$x_2 = 0, x_1 = 1$

Maximize $Z = 9 + 6x_3 + 4x_4$

Subject to

Subject to:

$5x_3 + 2x_4 \leq 4$

$x_3 \leq 1$

$x_4 \leq 0$

$x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 3,4$

$x_2 = 1, x_1 = 1$

Maximize $Z = 9 + 5 + 6x_3 + 4x_4$

Subject to:

$5x_3 + 2x_4 \leq 1$

$x_3 \leq 1$

$x_4 \leq 1$

$x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 3,4$

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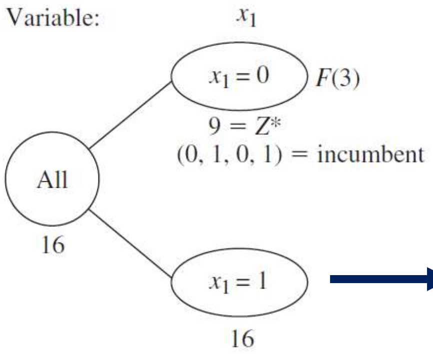
- Continuing the example



$x_2 = 0, x_1 = 1$
Maximize $Z = 9 + 6x_3 + 4x_4$
Subject to
Subject to:
$5x_3 + 2x_4 \leq 4$
$x_3 \leq 1$
$x_4 \leq 0$
$x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 3,4$

$x_2 = 1, x_1 = 1$
Maximize $Z = 9 + 5 + 6x_3 + 4x_4$
Subject to:
$5x_3 + 2x_4 \leq 1$
$x_3 \leq 1$
$x_4 \leq 1$
$x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 3,4$

Linear programming applied to these solutions yields

$x_1, x_2, x_3, x_4 = \left(1,0,\frac{4}{5},0\right)$ with $Z = 13.8$

$x_1, x_2, x_3, x_4 = \left(1,1,0,\frac{1}{2}\right)$ with $Z = 16$

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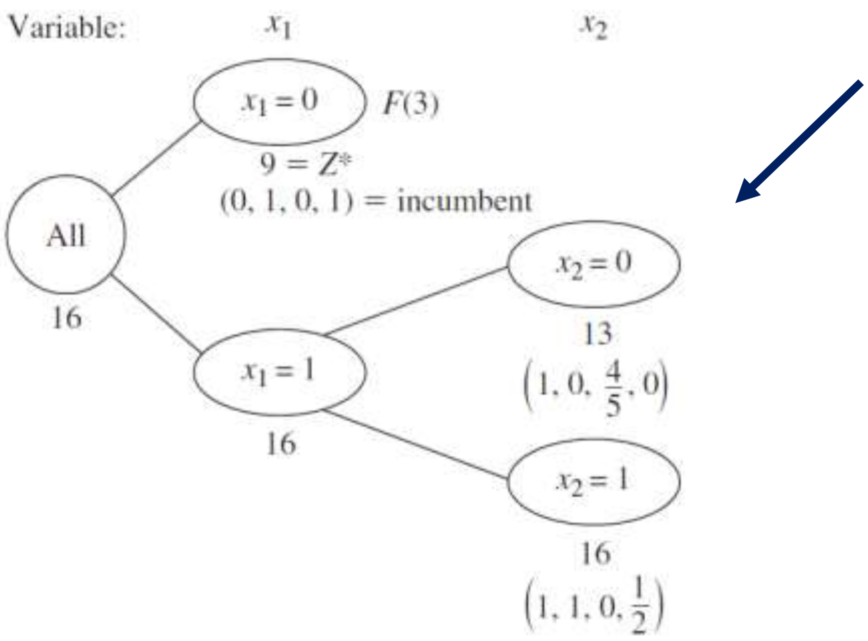
- Continuing the example



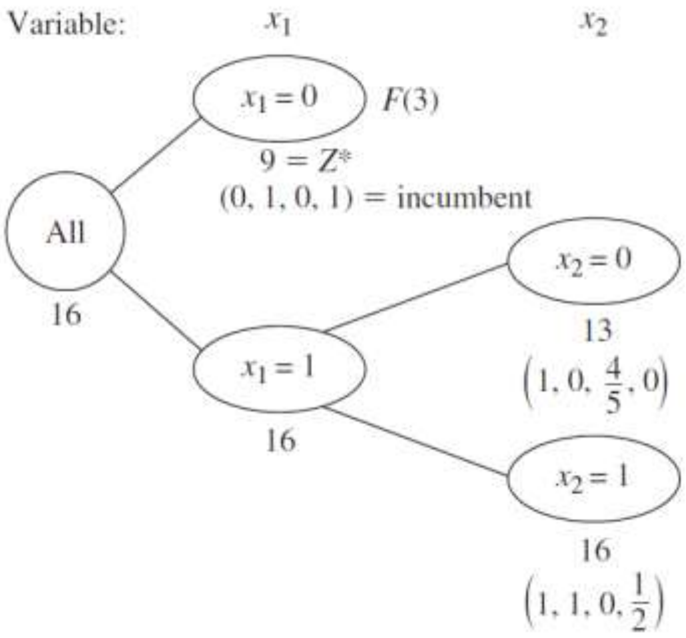
This is where we are now; no problem has been bound or fathomed at this step

Test 1: Its bound $\leq Z^*$ NO

Test 2: Its LP relaxation has no feasible solutions NO

Test 3: The optimal solution for its LP relaxation is integer NO

- Continuing the example



Since the problem $x_2 = 1$ has the larger Z we branch this solution splitting on x_3

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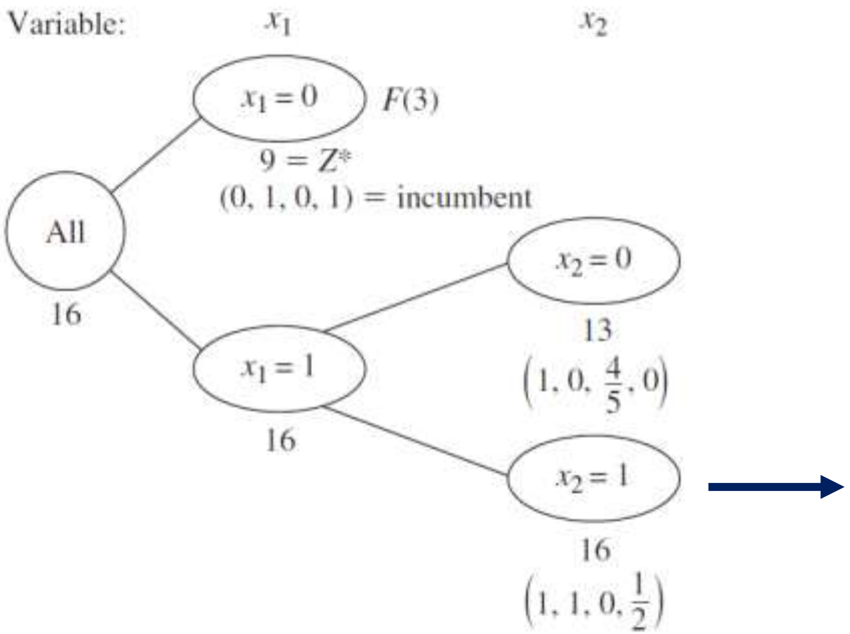
- Continuing the example; note how both Z and the constraints change to adopt to the new values



$x_3 = 0,\ x_1 = 1,\ x_2 = 1$

Maximize $Z = 14 + 4x_4$

Subject to:

$2x_4 \leq 1$ This was $5x_3 + 2x_4 \leq 1$

$x_4 \leq 1$

$x_4 \leq 1$ This was $x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 4$

$x_3 = 1,\ x_1 = 1,\ x_2 = 1$

Maximize $Z = 20 + 4x_4$

Subject to:

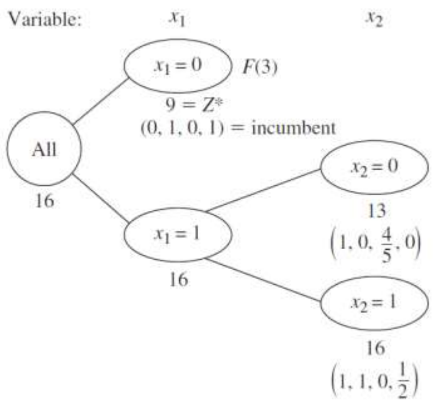
$2x_4 \leq -4$ This was $5x_3 + 2x_4 \leq 1$

$x_4 \leq 1$

$x_4 \leq 0$ This was $x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 4$

- Continuing the example



$x_3 = 0, x_1 = 1,\ x_2 = 1$
Maximize $Z = 14 + 4x_4$
Subject to:
$2x_4 \le 1$
$x_4 \le 1$
$x_4 \le 1$
$x_j \ge 0$ for $j = 4$

$x_3 = 1, x_1 = 1,\ x_2 = 1$
Maximize $Z = 20 + 4x_4$
Subject to:
$2x_4 \le -4$
$x_4 \le 1$
$x_4 \le 0$
$x_j \ge 0$ for $j = 4$

Linear programming applied to these solutions yields no feasible integer solution

$x_1, x_2, x_3, x_4 = \left(1,1,0,\frac{1}{2}\right)$ with $Z = 16$

$x_1, x_2, x_3, x_4 =$ no feasible solution

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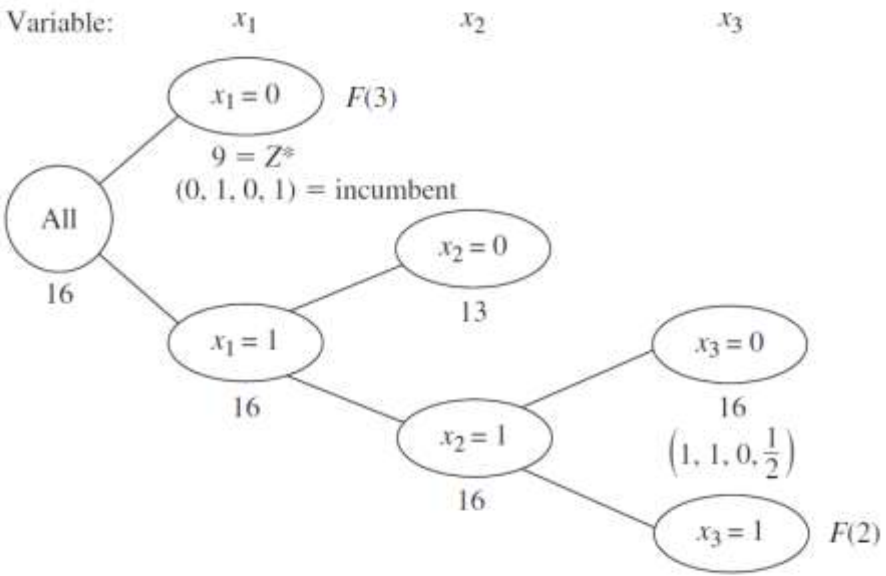
- Continuing the example

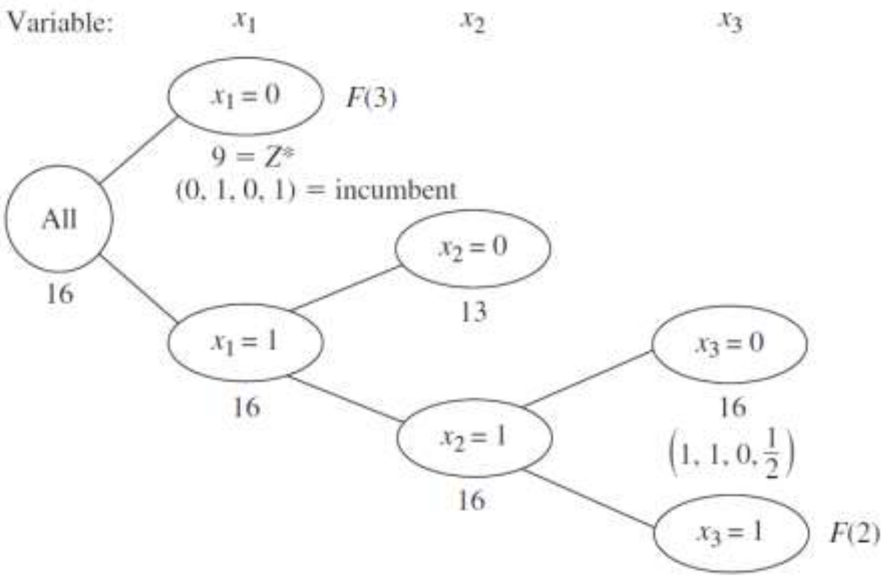


This is where we are now, with one solution fathomed and one open

No test failed

Test 2 failed

- Continuing the example)

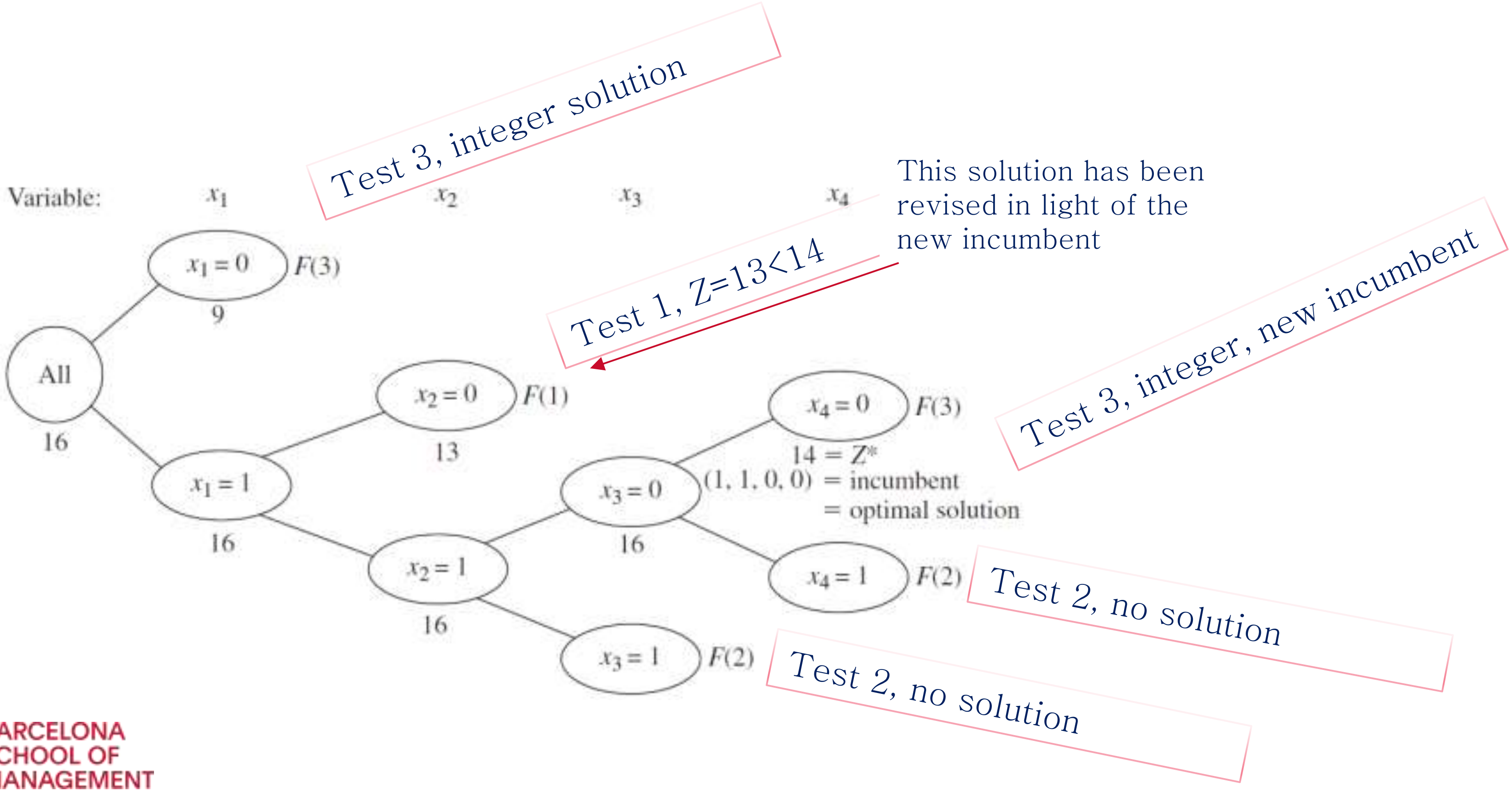


We now branch the problem with $x_3 = 0$, but since only variable x_4 is left fixing it generates directly a solution!

For $x_4 = 0$
$x_1, x_2, x_3, x_4 = (1{,}1{,}0{,}0)$ with $Z = 14$

For $x_4 = 1$
$x_1, x_2, x_3, x_4 = (1{,}1{,}0{,}1)$ unfeasible

- Continuing the example



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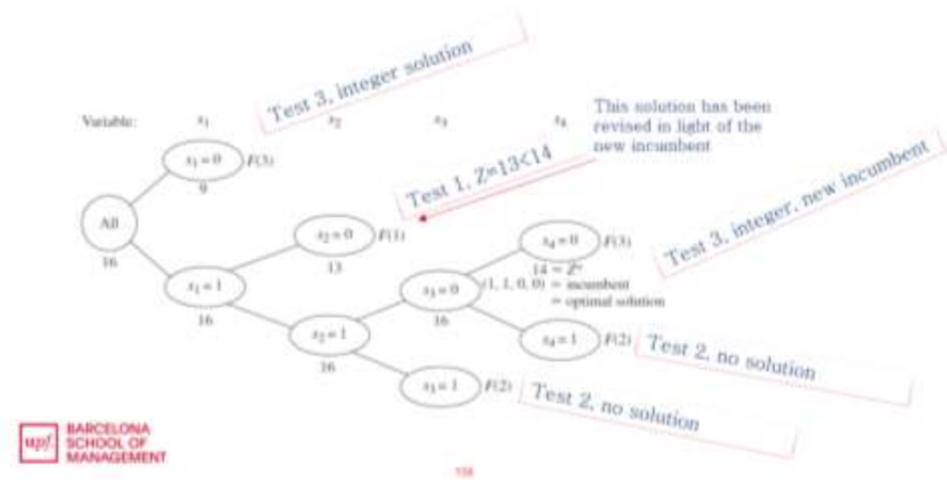

The solution is laborious, Needs book-keeping of how objective and constraints change in the various branches, and repeated recourse to LP, simplex calculations

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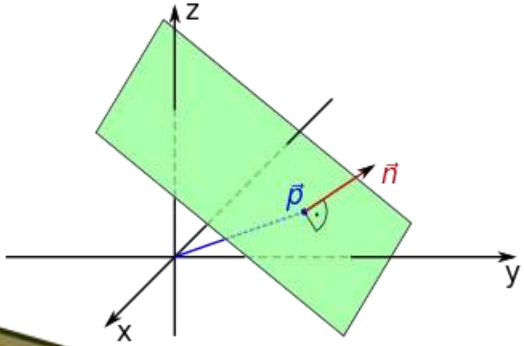

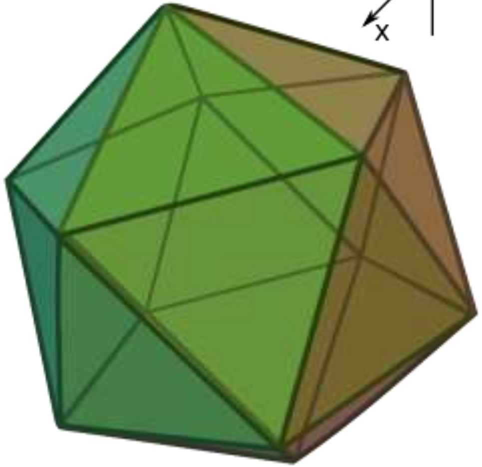

Source (both images): Wikipedia Commons

Some take home points

Integer programming and linear programming: LP=convex polyhedron touched by the hyperplane of the objective function; the IP solutions instead are isolated point inside the polyhedron

Find these points may not be easy but the LP solution is an upper bound for the Z of IP

Panettone with raisins inside

Source: https://leitesculinaria.com/478/recipes-cranberry-pistachio-panettone.html

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Homework

1) Write the equations for this transportation problem knowing that freight cost for each shipment is $100 plus 50 cents per mile. How much should be shipped from each plant to each of the distribution centers to minimize the total shipping cost?

9.1-2. The Childfair Company has three plants producing child push chairs that are to be shipped to four distribution centers. Plants 1, 2, and 3 produce 12, 17, and 11 shipments per month, respectively. Each distribution center needs to receive 10 shipments per month. The distance from each plant to the respective distributing centers is given below:

		Distance			
		Distribution Center			
		1	2	3	4
Plant	1	800 miles	1,300 miles	400 miles	700 miles
	2	1,100 miles	1,400 miles	600 miles	1,000 miles
	3	600 miles	1,200 miles	800 miles	900 miles

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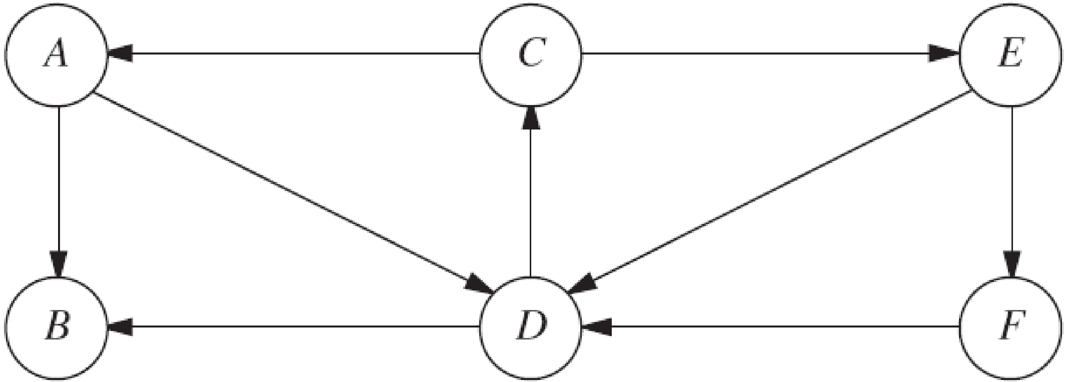
Homework

2) Consider the following directed network (Hillier 10.2-1)



(a) Find a directed path from node A to node F, and then identify three other undirected paths from node A to node F.
(b) Find three directed cycles. Then identify an undirected cycle that includes every node.
(c) Identify a set of arcs that forms a spanning tree.
(d) Use the process illustrated in Fig. 10.3 to grow a tree one arc at a time until a spanning tree has been formed. Then repeat this process to obtain another spanning tree. [Do not duplicate the spanning tree identified in part (c).]

Homework 3) You need to take a trip by car to another town that you have never visited before. Therefore, you are studying a map to determine the shortest route to your destination. Depending on which route you choose, there are five other towns (call them A, B, C, D, E) that you might pass through on the way. The map shows the mileage along each road that directly connects two towns without any intervening towns. These numbers are summarized in the following table, where a dash indicates that there is no road directly connecting these two towns without going through any other towns. Formulate this problem as a shortest-path problem by drawing a network where nodes represent towns, links represent roads, and numbers indicate the length of each link in miles.

	Miles between Adjacent Towns					
Town	**A**	**B**	**C**	**D**	**E**	**Destination**
Origin	40	60	50	—	—	—
A		10	—	70	—	—
B			20	55	40	—
C				—	50	—
D					10	60
E						80

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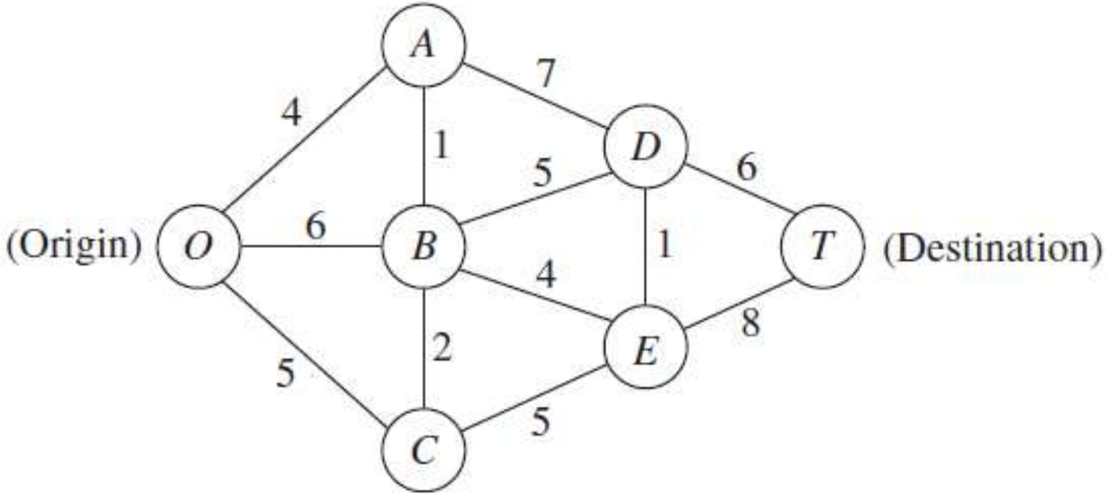
Homework **4)** Find shortest path from O to T, first visually then using the table method and backward recursion studied in Lesson 4 (Hillier 10.3-4); the first row of the table in give below.



n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	nth Nearest Node	Minimum Distance	Last Connection
1	O	A	4	A	4	OA

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Thank you

www.andreasaltelli.eu
https://orcid.org/0000-0003-4222-6975
@AndreaSaltelli@mstdn.social
https://www.youtube.com/channel/UCz26ZK04xchekUy4GevA3DA

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