Máster Universitario en Administración y Dirección de Empresas
Full Time MBA

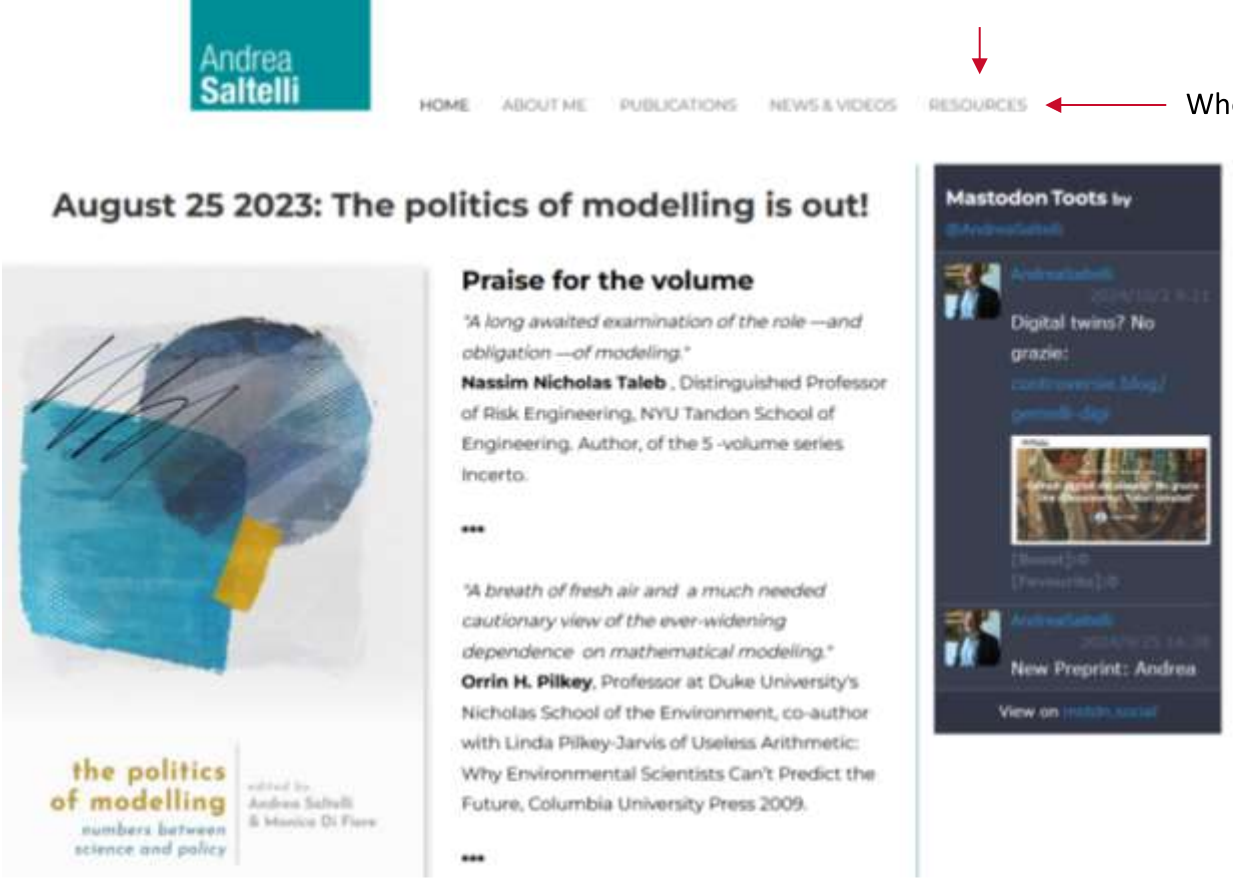
Quantitative methods for decision making

Professor Andrea Saltelli

BARCELONA SCHOOL OF MANAGEMENT

Elements of quantification for decision making with emphasis on operation research

BARCELONA SCHOOL OF MANAGEMENT



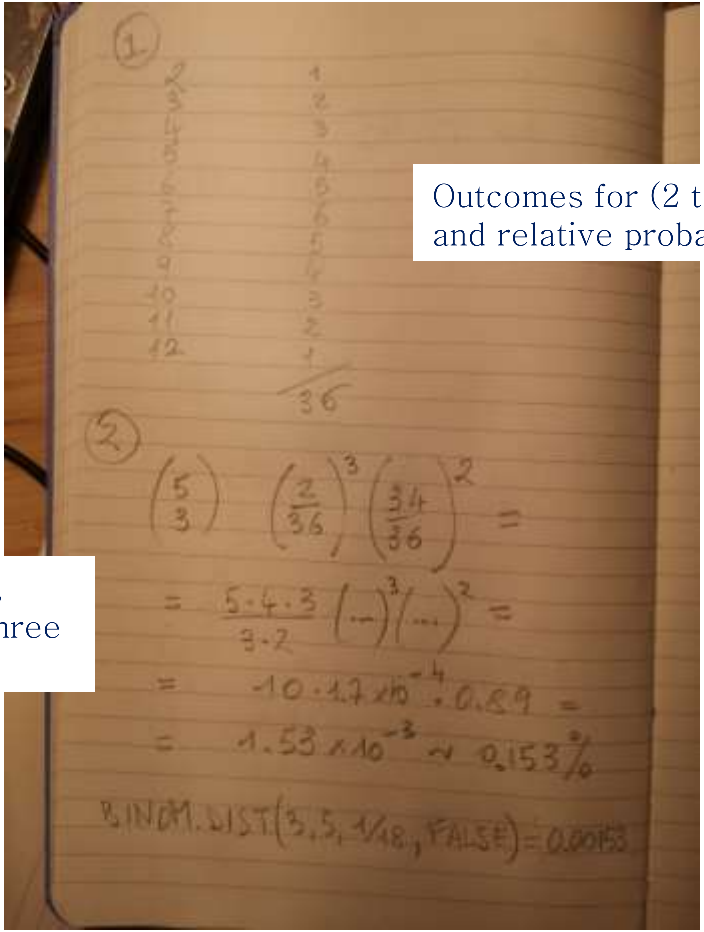
Where to find this talk

The talk is also at

https://ecampus.bsm.upf.edu/,

where you find additional reading material

Solutions homework Lesson 1



Outcomes for (2 to 12) and relative probability

Five throws, making 11 three times

Some more stats
(Mann, p. 150)

Suppose all 100 employees of a company were asked whether they are in favor of or against paying high salaries to CEOs of U.S. companies. Table 4.3 gives a two-way classification of the responses of these 100 employees.

Table 4.4 Two-Way Classification of Employee Responses with Totals

	In Favor	Against	Total
Male	15	45	60
Female	4	36	40
Total	19	81	100

Contingency table

Suppose all 100 employees of a company were asked whether they are in favor of or against paying high salaries to CEOs of U.S. companies. Table 4.3 gives a two-way classification of the responses of these 100 employees.

Table 4.4 Two-Way Classification of Employee Responses with Totals

	In Favor (A)	Against (B)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100

Suppose all 100 employees of a company were asked whether they are in favor of or against paying high salaries to CEOs of U.S. companies. Table 4.3 gives a two-way classification of the responses of these 100 employees.

Table 4.4 **Two-Way Classification of Employee Responses with Totals**

	In Favor (A)	Against (B)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100

We randomly extract one name out of the 100. If we call 'in favour' A and 'against' B, then:

What are

P(A), P(F),
P(M), P(F)
P(A∩F)

If one name is extracted and it is a woman, then:

What is P(A|F)

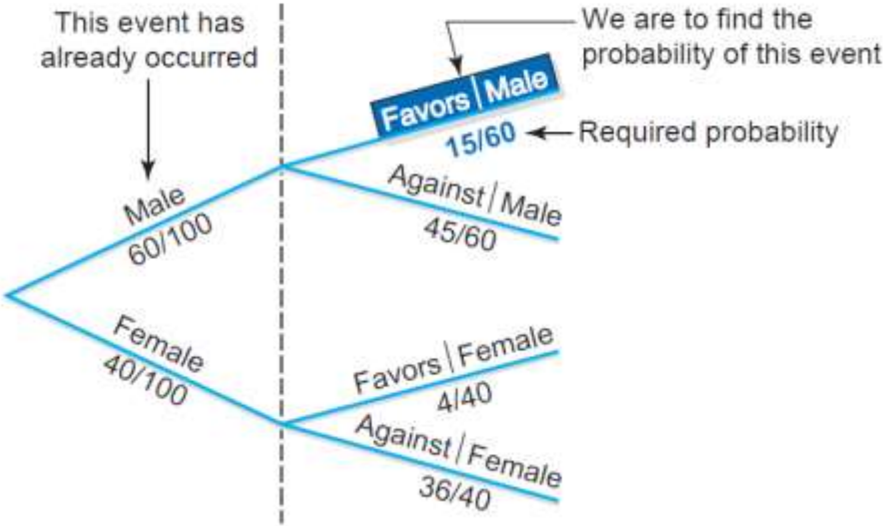
If one name is extracted and it is a in favour, then:

What is P(F|A)

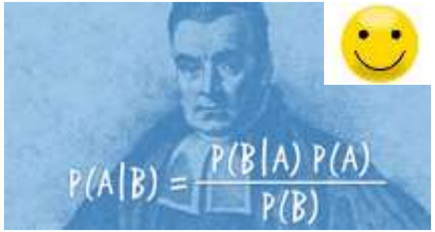
Suppose all 100 employees of a company were asked whether they are in favor of or against paying high salaries to CEOs of U.S. companies. Table 4.3 gives a two-way classification of the responses of these 100 employees.

Table 4.4 **Two-Way Classification of Employee Responses with Totals**

	In Favor (A)	Against (B)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100



$P(A \cap F) = P(A|F)P(F) = P(F|A)P(A)$ What is this formula?



$4/100 = (4/40)(40/100) = (4/19)(19/100)$ Is it verified?

In the next slides:

04 What is Operation Research?

05 A prototype example

06 Assumption of linear programming

07 More examples

08 Method of simplex

upf. BARCELONA SCHOOL OF MANAGEMENT

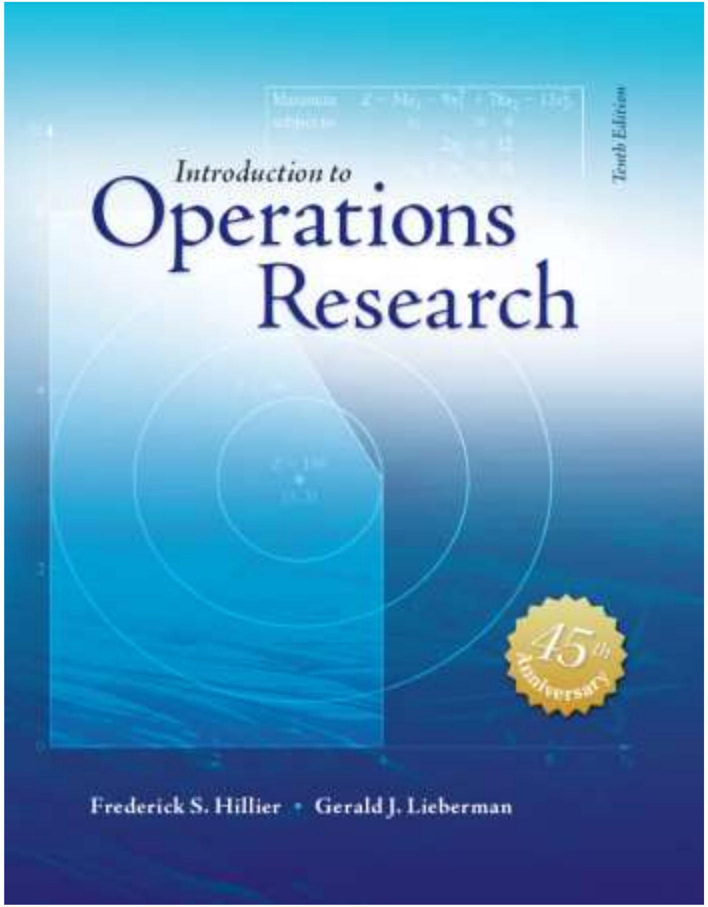
4.

What is Operation Research?

OR versus business analytics; some definitions; steps of an analysis; objectives, context and purpose; linear programming with examples and some theory. Hillier (10th edition, 2014) chapters 1 and 2.

Where to find this book:

https://www.andreasaltelli.eu/file/repository/Hillier_Ninth_Edition_Introducti.pdf



BARCELONA SCHOOL OF MANAGEMENT

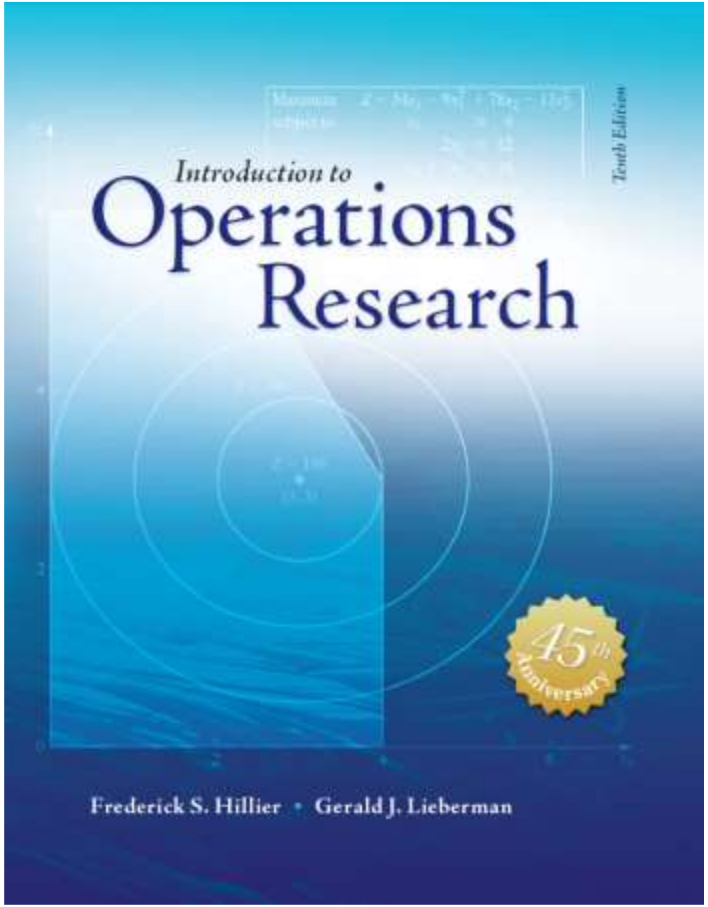
Operation Research (**OR**), Management Science, Analytics, business analytics:

What is the difference?

OR: “how to conduct and coordinate the operations (i.e. the activities) within an organization” (Hillier, p. 2)

> **OR** is research on operations applying the scientific method – foremost modelling and optimization.

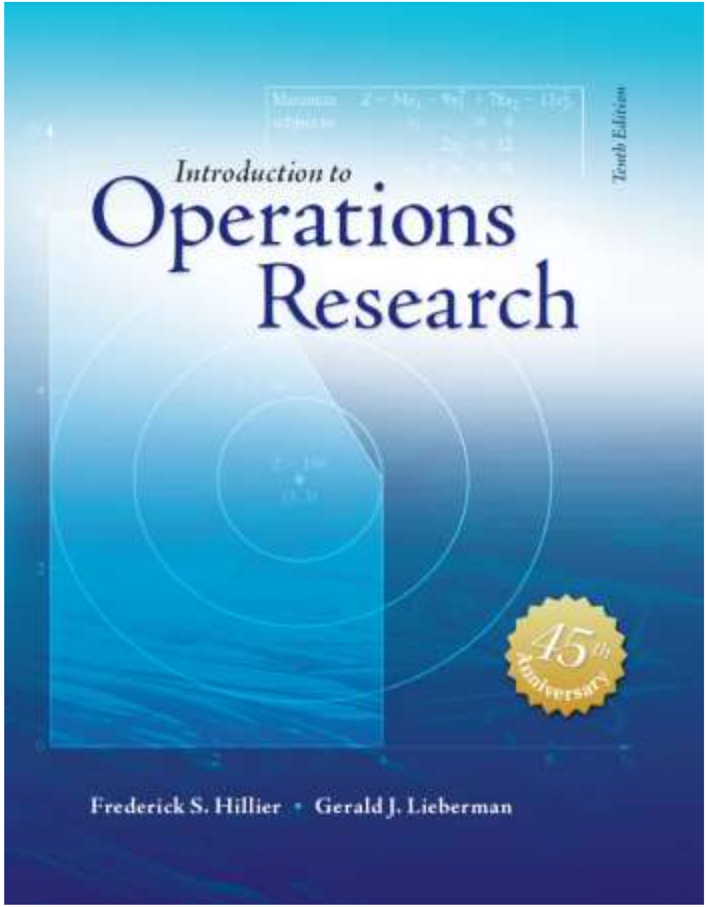
?



upf. BARCELONA SCHOOL OF MANAGEMENT

OR is research on operations applying the scientific method – foremost modelling and optimization

Modelling in OR is to be understood in very general terms, e.g. both mathematical and statistical



upf. BARCELONA SCHOOL OF MANAGEMENT

Operation Research, Management Science, Analytics, business analytics;

What is the difference?

"The term management science sometimes is used as a synonym for operations research"

How about "Analytics" (or Business Analytics)? Operation Research by another name as well?

upf. BARCELONA SCHOOL OF MANAGEMENT

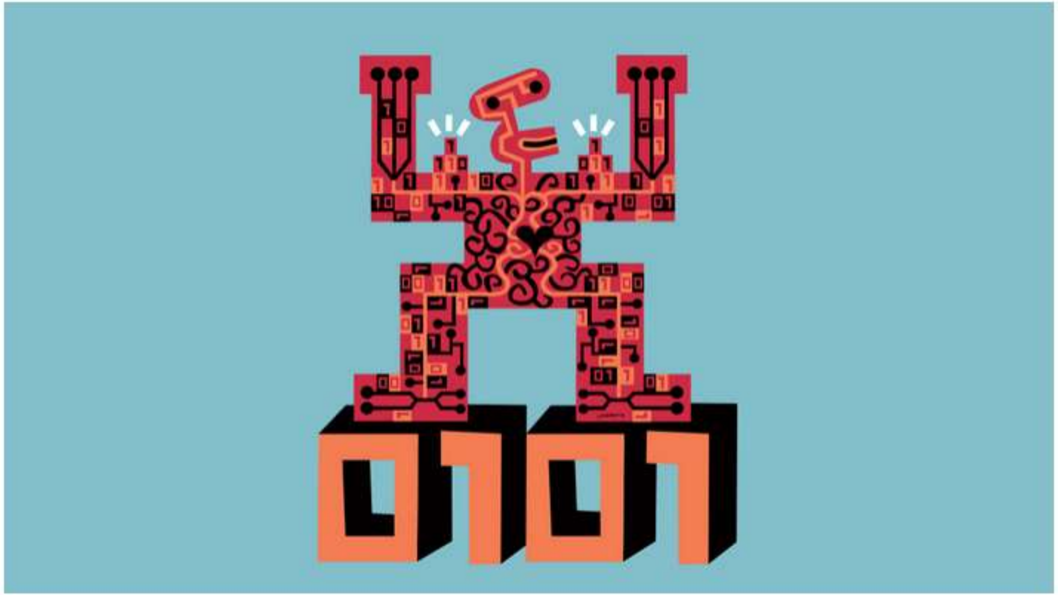
Analytics And Data Science

Competing on Analytics

Some companies have built their very businesses on their ability to collect, analyze, and act on data. Every company can learn from what these firms do. by Thomas H. Davenport

From the Magazine (January 2006)

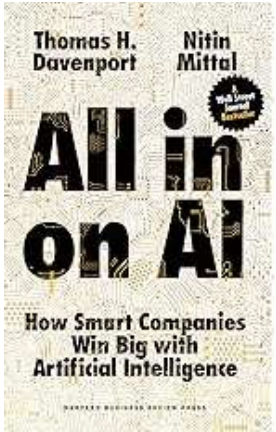
Harvard Business Review



Source: https://hbr.org/2006/01/competing-on-analytics; article open access here: https://www.researchgate.net/publication/7327312_Competing_on_Analytics

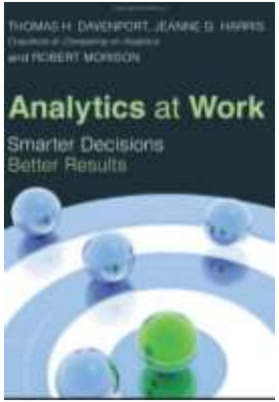
https://ecampus.bsm.upf.edu/

2023



2017

2010

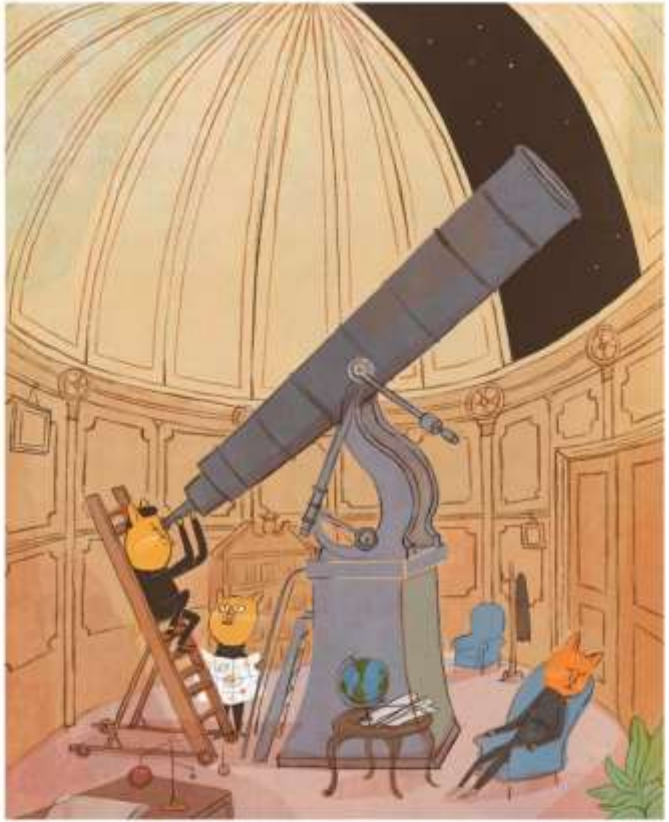


upf. BARCELONA SCHOOL OF MANAGEMENT

Business Analytics = Operation Research + big data

Analytics = scientific process of transforming data into insight for making better decisions

- **Descriptive** analytics, discover patterns e.g. via data mining
- **Predictive** analytics, use data to predict the future
- **Prescriptive** analytics, use data to guide present and future actions



Source: Tor Freeman, http://tormalore.blogspot.com/

upf. BARCELONA SCHOOL OF MANAGEMENT

Analytics 3.0: three analytics maturity levels

Analytics 1.0 organizations rely on internal data for decision making, rather than mere intuition

Analytics 2.0 companies combine internal data with externally sourced data, offering predictive capabilities

Analytics 3.0 firms actively generate data trails that can be collected and subsequently analysed

Source: https://hbr.org/2013/12/analytics-30

BARCELONA SCHOOL OF MANAGEMENT

Analytics 3.0: three analytics maturity levels

"Today it's not just information firms and online companies that can create products and services from analyses of data. It's every firm in every industry."

"The Bosch Group, based in Germany, is 127 years old, … has embarked on … intelligent fleet management, intelligent vehicle-charging infrastructures, intelligent energy management, intelligent security video analysis, and many more."

Harvard Business Review

Analytics And Data Science

Analytics 3.0

by Thomas H. Davenport

From the Magazine (December 2013)

Source: https://hbr.org/2013/12/analytics-30

Analytics 3.0: three analytics maturity levels

"Google, LinkedIn, Facebook, Amazon, and others have prospered not by giving customers information but by giving them **shortcuts to decisions and actions.**"

Harvard Business Review

Analytics And Data Science

Analytics 3.0

by Thomas H. Davenport

From the Magazine (December 2013)

Source: https://hbr.org/2013/12/analytics-30

Davenport's word of caution

"The use of prescriptive analytics often requires changes in the way frontline workers are managed …employees wearing or carrying sensors … Just as analytics that are intensely revealing of customer behavior have a certain "creepiness" factor, overly detailed reports of employee activity can cause discomfort. In the world of Analytics 3.0, there are times we need to look away."

Harvard Business Review

Analytics And Data Science

Analytics 3.0

by Thomas H. Davenport

From the Magazine (December 2013)

Source: https://hbr.org/2013/12/analytics-30

https://ecampus.bsm.upf.edu/

A critical angle: Teachout, Z. (2022). The Boss Will See You Now | Zephyr Teachout. New York Review of Books. https://www.nybooks.com/articles/2022/08/18/the-boss-will-see-you-now-zephyr-teachout/ available in eCampus

The New York Review of Books

upf. BARCELONA SCHOOL OF MANAGEMENT

Analytics 3.0 firms actively generate data trails that can be collected and subsequently analysed

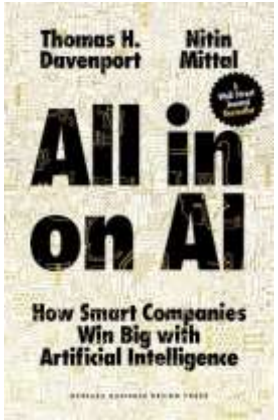
Do those books support platform capitalism? 👍 👎

➔ Platform capitalism

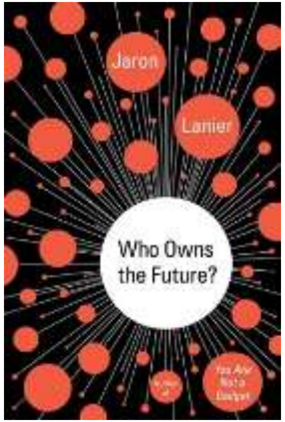
👍

👍

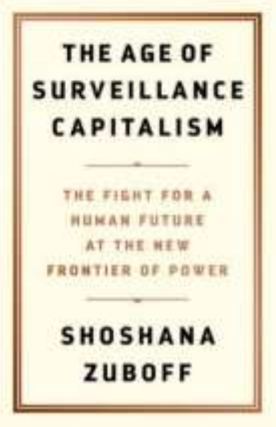
👍



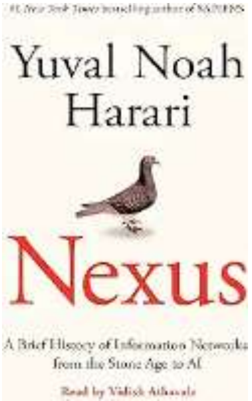
👎



👎

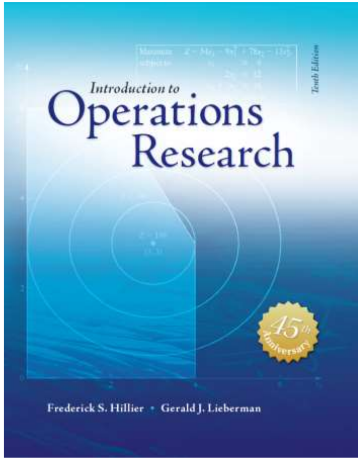


👎



upf. BARCELONA SCHOOL OF MANAGEMENT

1. Define the problem of interest and gather relevant data
2. Formulate a mathematical model to represent the problem.
3. Develop a computer-based procedure for deriving solutions to the problem from the model.
4. Test the model and refine it as needed.
5. Prepare for the ongoing application of the model as prescribed by management.
6. Implement (Hillier, p. 10)



upf. BARCELONA SCHOOL OF MANAGEMENT

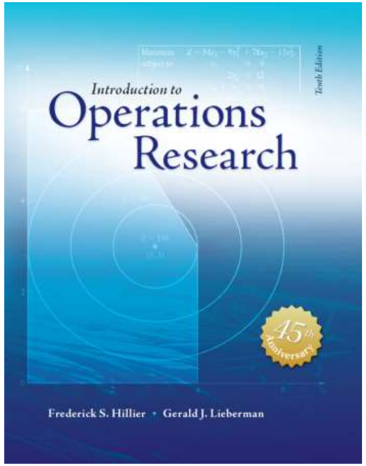
1. **Define the problem of interest and gather relevant data.**

2. Formulate a mathematical model to represent the problem.

Asymmetry of knowledge between owners of the problem and analysts

Purpose and context

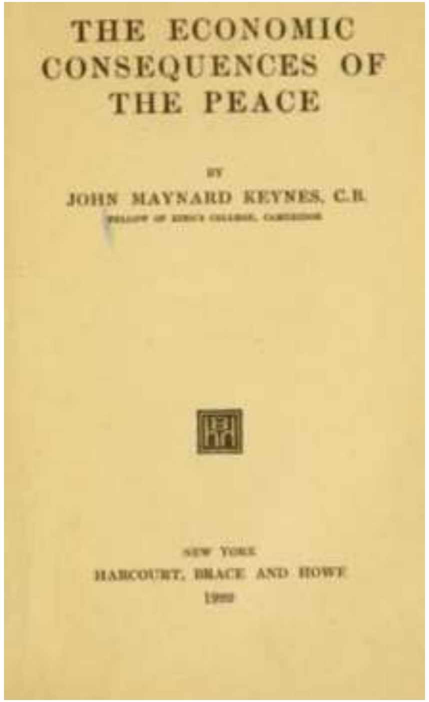
The definition of objectives



upf. BARCELONA SCHOOL OF MANAGEMENT

1. **Define the problem of interest and gather relevant data.**

2. Formulate a mathematical model to represent the problem.



John Maynard Keynes

Said (possibly) by? "Better to be roughly right than precisely wrong"

...possibly the best cost benefit analysis ever ... another precursor of OR?

"Better be roughly right than precisely wrong"

"Lack of mathematical culture is revealed nowhere so conspicuously, as in meaningless precision in numerical computations" (Carl Friedrich Gauss)

https://www.bluecrux.com/blog/why-mr-spock-would-never-make-a-good-planner/

upf. BARCELONA SCHOOL OF MANAGEMENT

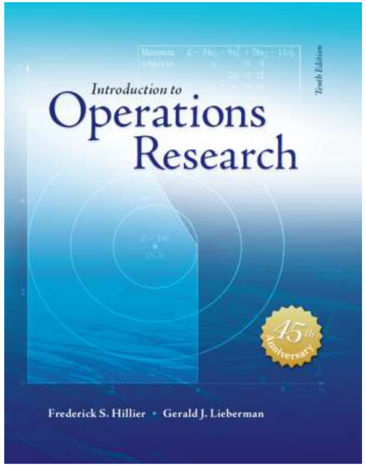
Objectives of the analysis : There are responsibilities beyond maximization of objectives

Carroll AB. The Pyramid of Corporate Social Responsibility: Toward the moral management of organizational stakeholders. 1991; Business Horizons, **34**(4), July-August:39–48. Source: https://www.financialeducatorscouncil.org/corporate-social-responsibility-definition-and-history/

upf. BARCELONA SCHOOL OF MANAGEMENT

Obligations toward

1. the owners (stockholders, etc.), who desire profits (dividends, stock appreciation, and so on);
2. the employees, who desire steady employment at reasonable wages;
3. the customers, who desire a reliable product at a reasonable price;
4. the suppliers, who desire integrity and a reasonable selling price for their goods; and
5. the government and hence the nation (Hillier, p. 12)



← Responsibilities beyond maximization of objectives

BARCELONA SCHOOL OF MANAGEMENT

Pitfalls in Formulation and Modelling

Box 3.1 Pitfalls in formulation and modelling

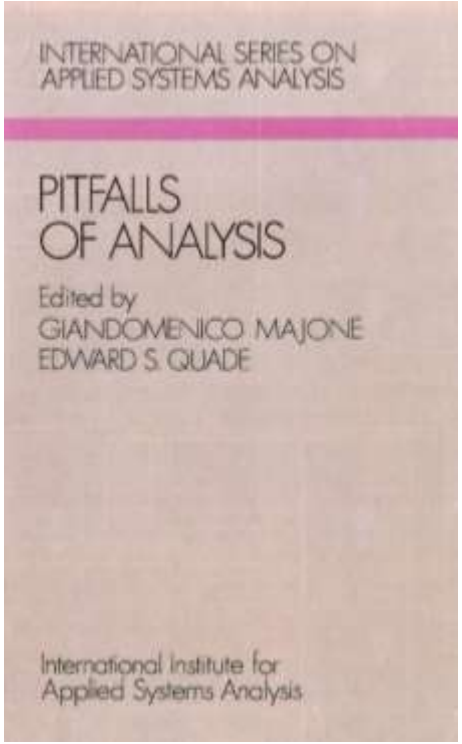
Pitfalls in formulation

Insufficient attention to formulation
Unquestioning acceptance of stated goals and constraints
Measuring achievement by proxy
Misjudging the difficulties
Bias

Pitfalls in modelling

Equating modelling with analysis
Improper treatment of uncertainties
Attempting to really simulate reality
Belief that a model can be proved correct
Neglecting the by-products of modelling
Overambition
Seeking academic rather than policy goals
Internalizing the policy maker
Not keeping the model relevant
Not keeping the model simple
Capture of the user by the modeller

Source: (Quade 1980)



https://ecampus.bsm.upf.edu/

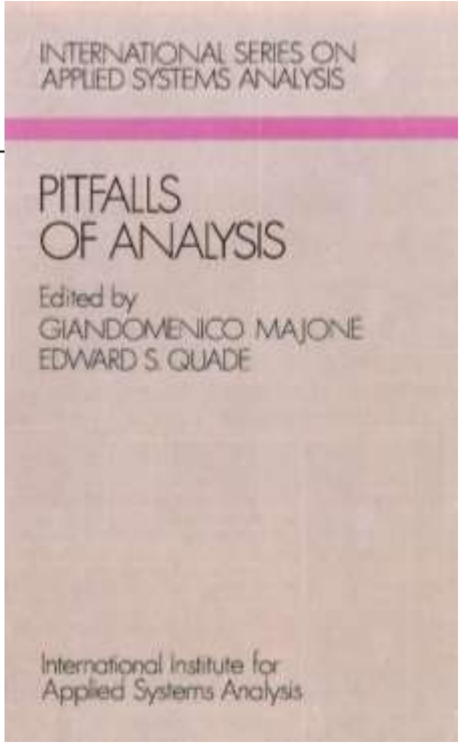
upf. BARCELONA SCHOOL OF MANAGEMENT

Pitfalls in Formulation and Modelling

Box 3.1 Pitfalls in formulation and modelling

Pitfalls in formulation

Insufficient attention to formulation
Unquestioning acceptance of stated goals and constraints
Measuring achievement by proxy
Misjudging the difficulties
Bias



BARCELONA SCHOOL OF MANAGEMENT

Comments here?

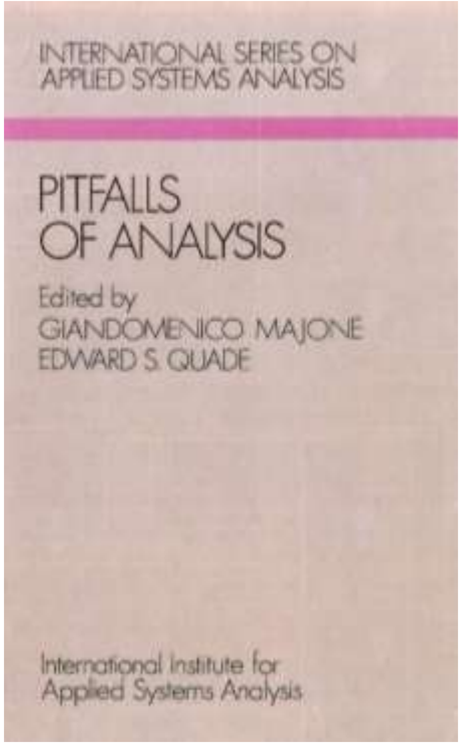
Pitfalls in Formulation and Modelling

Pitfalls in modelling

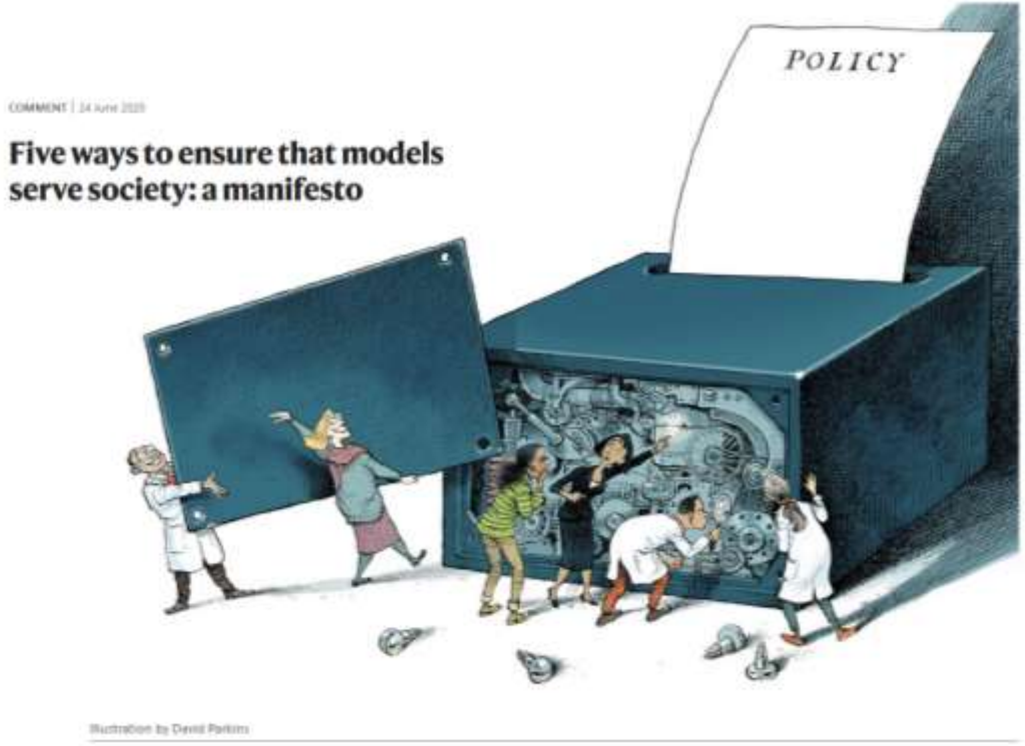
Equating modelling with analysis
Improper treatment of uncertainties
Attempting to really simulate reality
Belief that a model can be proved correct
Neglecting the by-products of modelling
Overambition
Seeking academic rather than policy goals
Internalizing the policy maker
Not keeping the model relevant
Not keeping the model simple
Capture of the user by the modeller

Source: (Quade 1980)

Pick up one!



BARCELONA SCHOOL OF MANAGEMENT



Available in eCampus

As modeller, beware your own bias

As a user, beware model seduction

BARCELONA SCHOOL OF MANAGEMENT

1. Define the problem of interest and gather relevant data.

2. Formulate a mathematical model to represent the problem.

3. **Develop a computer-based procedure for deriving solutions to the problem from the model.**

4. Test the model and refine it as needed.

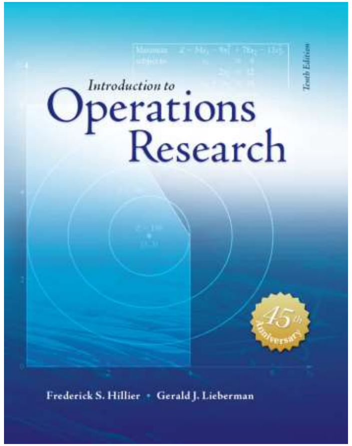
5. Prepare for the ongoing application of the model as prescribed by management.

6. Implement (Hillier, p. 10)

Seek ‘satisficing’ solutions (satisfy + suffice)

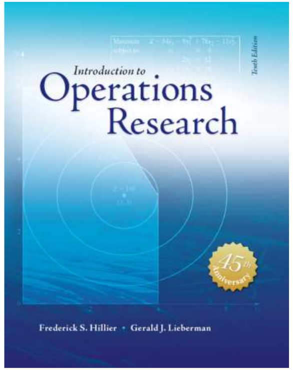
Post-optimality analysis
What-if analysis

Uncertainty and sensitivity analysis



upf. BARCELONA SCHOOL OF MANAGEMENT

1. Define the problem of interest and gather relevant data.
2. Formulate a mathematical model to represent the problem.
3. Develop a computer-based procedure for deriving solutions to the problem from the model.
4. **Test the model and refine it as needed.**
5. Prepare for the ongoing application of the model as prescribed by management.
6. Implement (Hillier, p. 10)

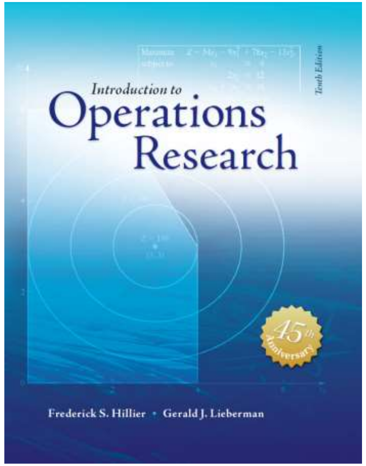


Interactive tools to make allowance for revisions;

More sensitivity & uncertainty analysis

upf. BARCELONA SCHOOL OF MANAGEMENT

1. Define the problem of interest and gather relevant data.
2. Formulate a mathematical model to represent the problem.
3. Develop a computer-based procedure for deriving solutions to the problem from the model.
4. Test the model and refine it as needed.
5. **Prepare for the ongoing application of the model as prescribed by management.**
6. **Implement** (Hillier, p. 10)



Documentation

Replicability, reproducibility

upf. BARCELONA SCHOOL OF MANAGEMENT

5.

A prototype example

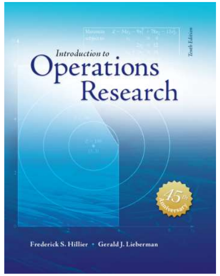
An example with most of the features of a linear programming setting. Hillier 2014, chapter 3.

BARCELONA SCHOOL OF MANAGEMENT

A typical linear programming setting: allocating limited resources among competing activities in a best possible (i.e., optimal) way: the WYNDOR GLASS CO. producing doors and windows

Tree plants. Aluminium frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products.

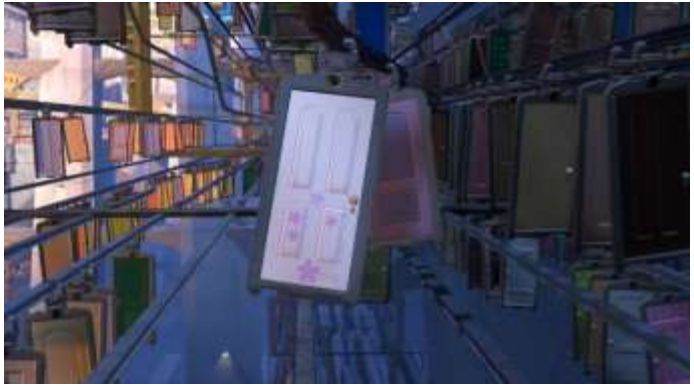
Source: PIXAIR’s Monsters and Co.



upf. BARCELONA SCHOOL OF MANAGEMENT

Two new products to be put into production:

Product 1: An 8-foot glass door with aluminium framing
Product 2: A 4 6 foot double-hung wood-framed window



Source: PIXAIR's Monsters and Co.

- Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2.
- Product 2 needs only Plants 2 and 3.

	Production Time per Batch, Hours	
	Product	
Plant	1	2
1	1	0
2	0	2
3	3	2

upf. BARCELONA SCHOOL OF MANAGEMENT

But time in the three plants is limited because of competing productions

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18

BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

And the profits per batch of product are different

upf. BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.1 Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

The key steps in formulating this as a linear programming problem are

- What objective needs maximizing/minimizing ← Easy, guess!
- What are the decision variables ← Less easy

upf. BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

Z =total profit per week in thousands of dollars from producing these batches

x_1 = number of batches per week of product 1 to be produced
x_2 = number of batches per week of product 2 to be produced ← One possibility

The decision variables are thus x_1 and x_2 and the objective to be maximized Z is:

$$Z = 3x_1 + 5x_2$$ ← The definition of objective depends upon the decision variable

BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

But production time per plant is limited:

From the rightmost column of the table

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

Done?

The model does not know yet that the numbers must be positive; thus:

$$x_1 \geq 0$$
$$x_2 \geq 0$$

upf. BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

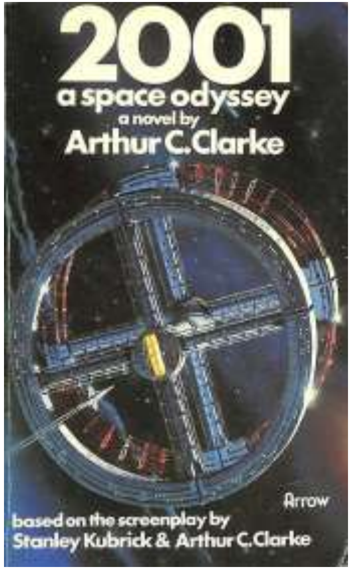
Maximize $Z = 3x_1 + 5x_2$

Subject to:

$$\begin{aligned} x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

A 'magic' conversion from a table of data to a set of equation…

"Any sufficiently advanced technology is indistinguishable from magic" (Arthur C. Clark)



upf. BARCELONA SCHOOL OF MANAGEMENT

■ **TABLE 3.1** Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

Maximize $Z = 3x_1 + 5x_2$

Subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

It is not difficult to imagine how one could get this magic wrong; e.g. define the decision variables as:

x_{1j} = number of batches per week of product 1 to be produced in plant j, $j = 1,2,3$

x_{2j} = number of batches per week of product 2 to be produced in plant j, $j = 1,2,3$

BARCELONA SCHOOL OF MANAGEMENT

Maximize $Z = 3x_1 + 5x_2$
Subject to:

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

Old equations

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

x_1 = number of batches per week of product 1 is replaced by x_{11}, x_{12}, x_{13}

x_2 = number of batches per week of product 2 is replaced by x_{21}, x_{22}, x_{23}

Try to write the new equations

Source: The Simpson, 20th Television Animation (The Walt Disney Company)

■ **TABLE 3.1** Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

x_{1j} = number of batches per week of product 1 to be produced in plant j

x_{2j} = number of batches per week of product 2 to be produced in plant j

$$Z = 3(x_{11}+x_{13}) + 5(x_{22} + x_{23})$$

$$x_{11} < 4$$

$$2x_{22} < 12$$

$$3x_{13} + 2x_{23} < 18$$

$$x_{11} \geq 0, \qquad x_{22} \geq 0$$

$$x_{13} \geq 0, \qquad x_{23} \geq 0$$

upf. BARCELONA SCHOOL OF MANAGEMENT

One way: Maximize $Z = 3x_1 + 5x_2$

Subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0, \qquad x_2 \geq 0$$

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Production Time per Batch, Hours — Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

The other way: Maximize $Z = 3(x_{11}+x_{13}) + 5(x_{22} + x_{23})$

Subject to:

$$x_{11} < 4$$
$$2x_{22} < 12$$
$$3x_{13} + 2x_{23} < 18$$
$$x_{11} \geq 0, \qquad x_{22} \geq 0$$
$$x_{13} \geq 0, \qquad x_{23} \geq 0$$

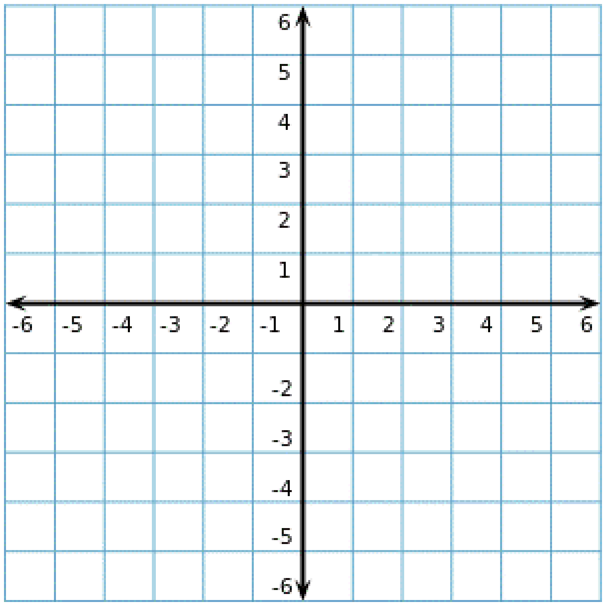
In what sense is this solution clumsier?

Source: Wikipedia Commons

..because of William of Occam (1287, 1347) and his razor

upf. BARCELONA SCHOOL OF MANAGEMENT

Since this problem is in two dimensions we can solve it graphically; back to Descartes, with his diagram



Source: https://study.com/learn/lesson/cartesian-coordinate-system.html

René Descartes
(1596-1650)

Maximize $Z = 3x_1 + 5x_2$

Subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

BARCELONA SCHOOL OF MANAGEMENT

Maximize $Z = 3x_1 + 5x_2$

Subject to:

$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

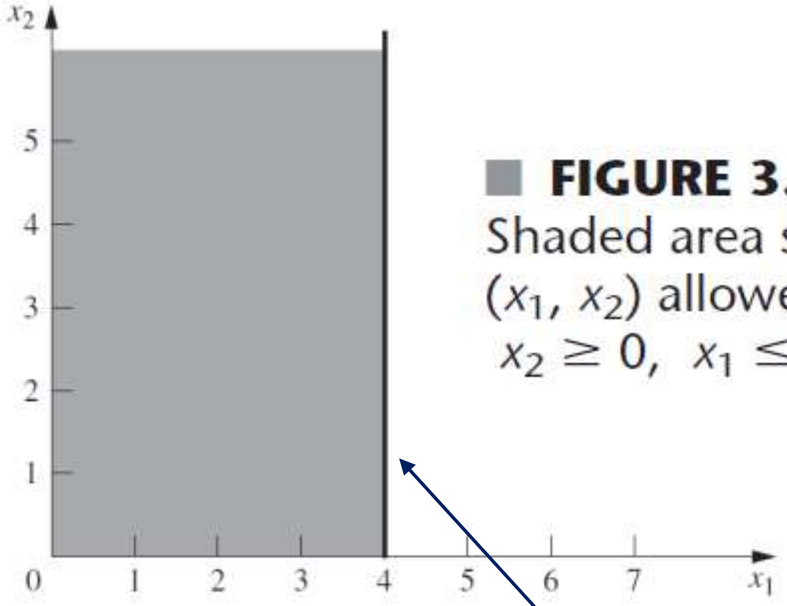


FIGURE 3.1
Shaded area shows values of (x_1, x_2) allowed by $x_1 \geq 0$, $x_2 \geq 0$, $x_1 \leq 4$.

Fist step: Draw a straight line following the equation $x_1 = 4$

BARCELONA SCHOOL OF MANAGEMENT

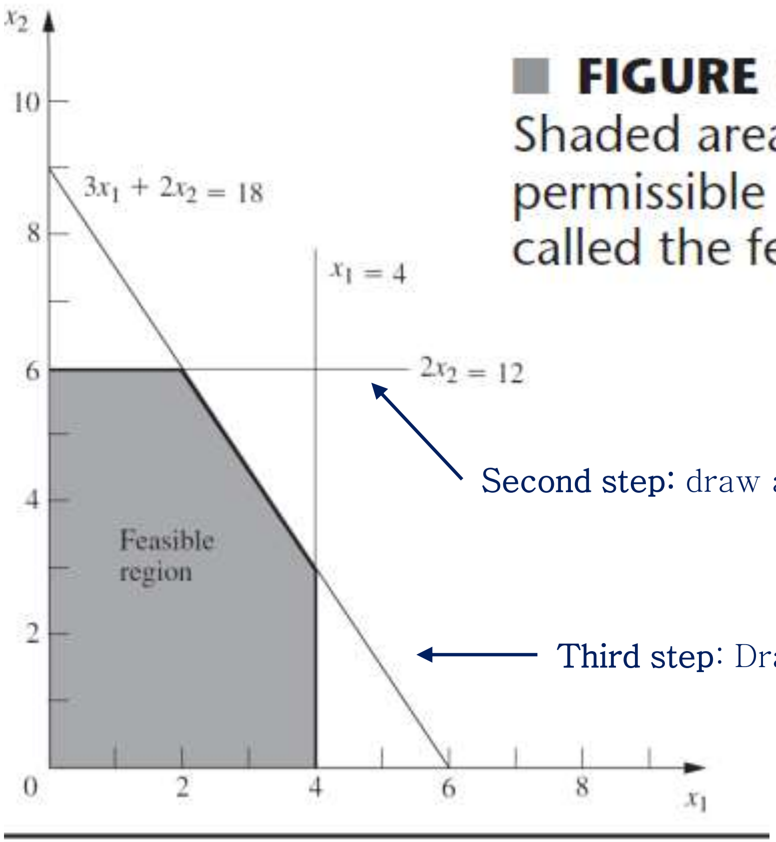


FIGURE 3.2
Shaded area shows the set of permissible values of (x_1, x_2), called the feasible region.

Maximize $Z = 3x_1 + 5x_2$

Subject to:

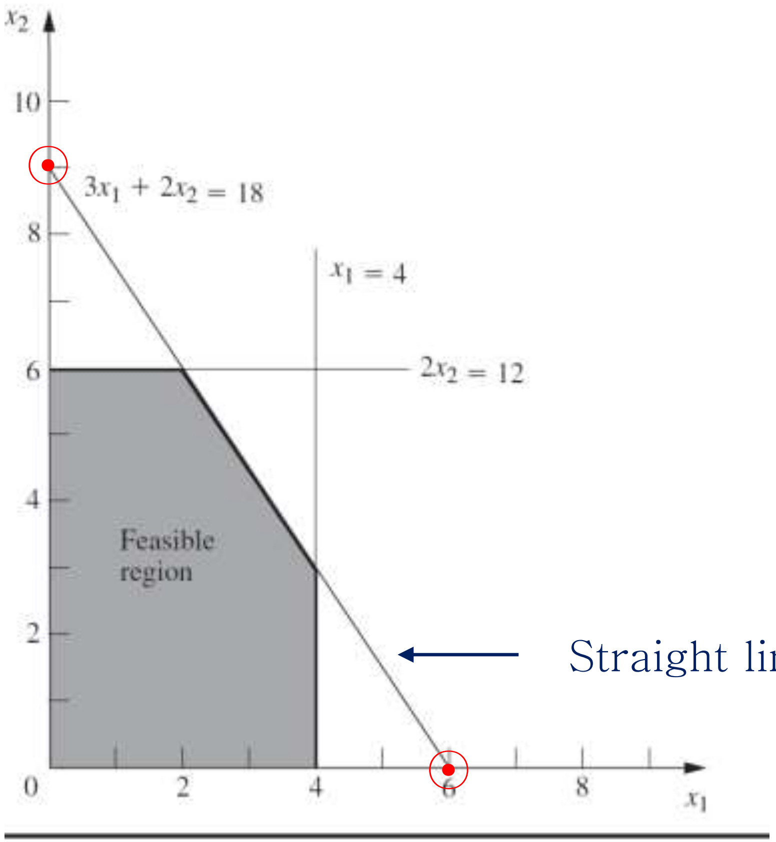
$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Second step: draw a straight line following the equation $2x_2 = 12$

Third step: Draw a straight line following the equation $3x_1 + 2x_2 = 18$

How?

BARCELONA SCHOOL OF MANAGEMENT



Tip to draw this line:

First, fix $x_1 = 0$
Plug it into $3x_1 + 2x_2 = 18$ to get $x_2 = 9$

Then, fix $x_2 = 0$
Plug it into $3x_1 + 2x_2 = 18$ to get $x_1 = 6$

➔ The line passes through points:

(x_1, x_2)=(0,9) and (x_1, x_2)= (6,0)

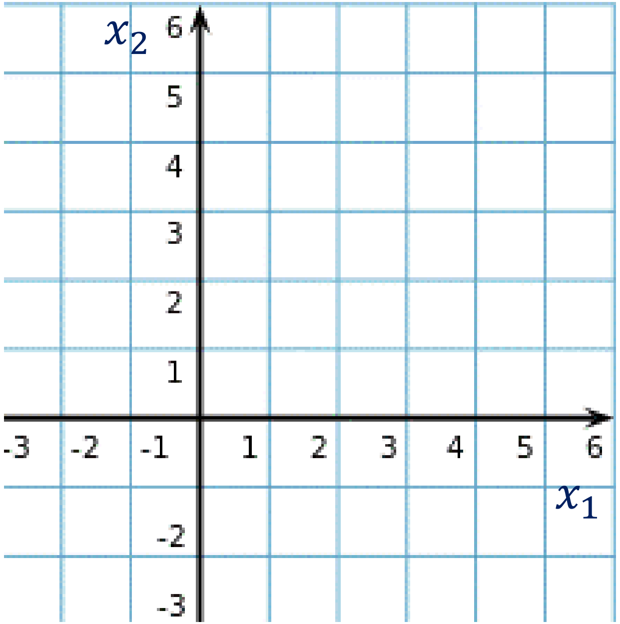
Straight line following the equation $3x_1 + 2x_2 = 18$

upf. BARCELONA SCHOOL OF MANAGEMENT

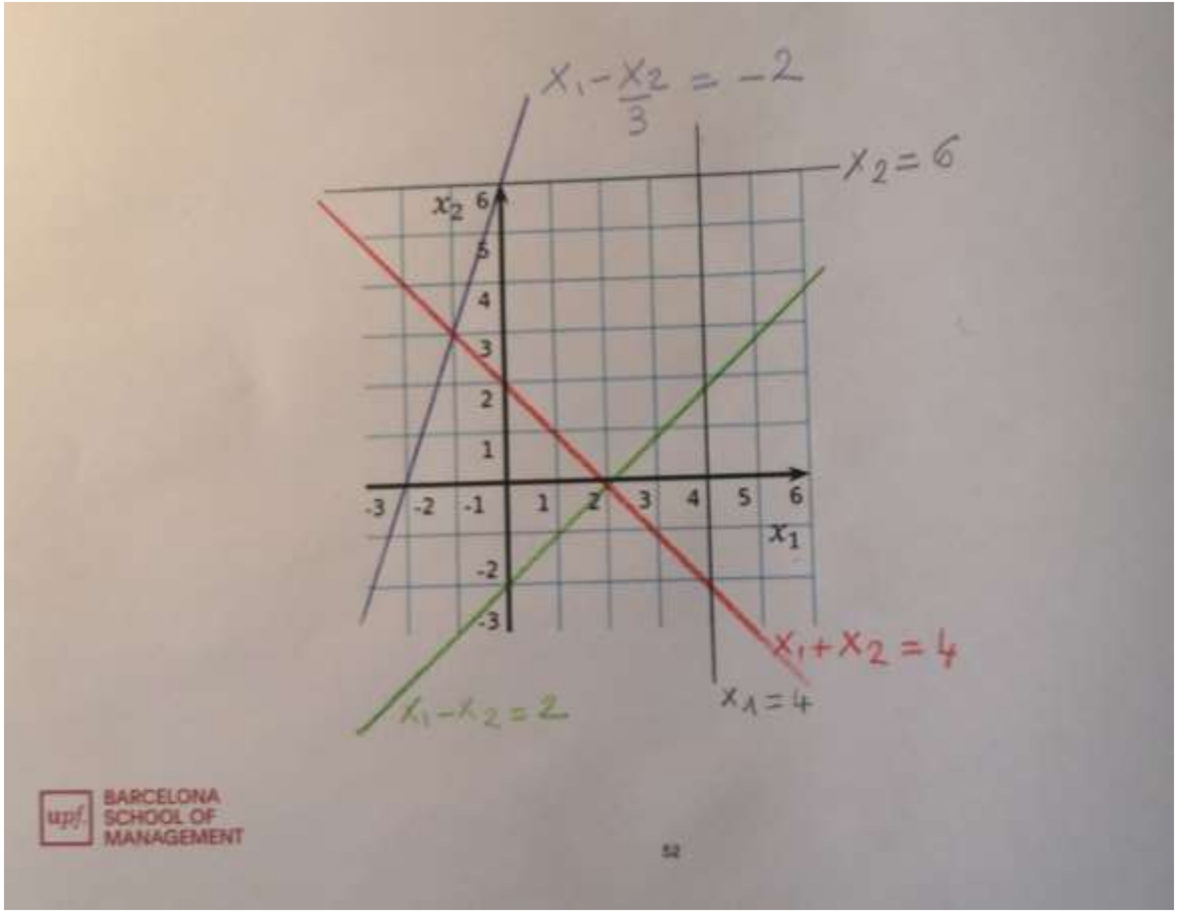
Paper, pencil ad ruler: please draw on a Cartesian diagram the straight lines

Source: The Simpson, 20th Television Animation (The Walt Disney Company)

$$x_1 = 4$$
$$x_2 = 6$$
$$x_1 + x_2 = 2$$
$$x_1 - x_2 = 2$$
$$x_1 - \frac{1}{3}x_2 = -2$$



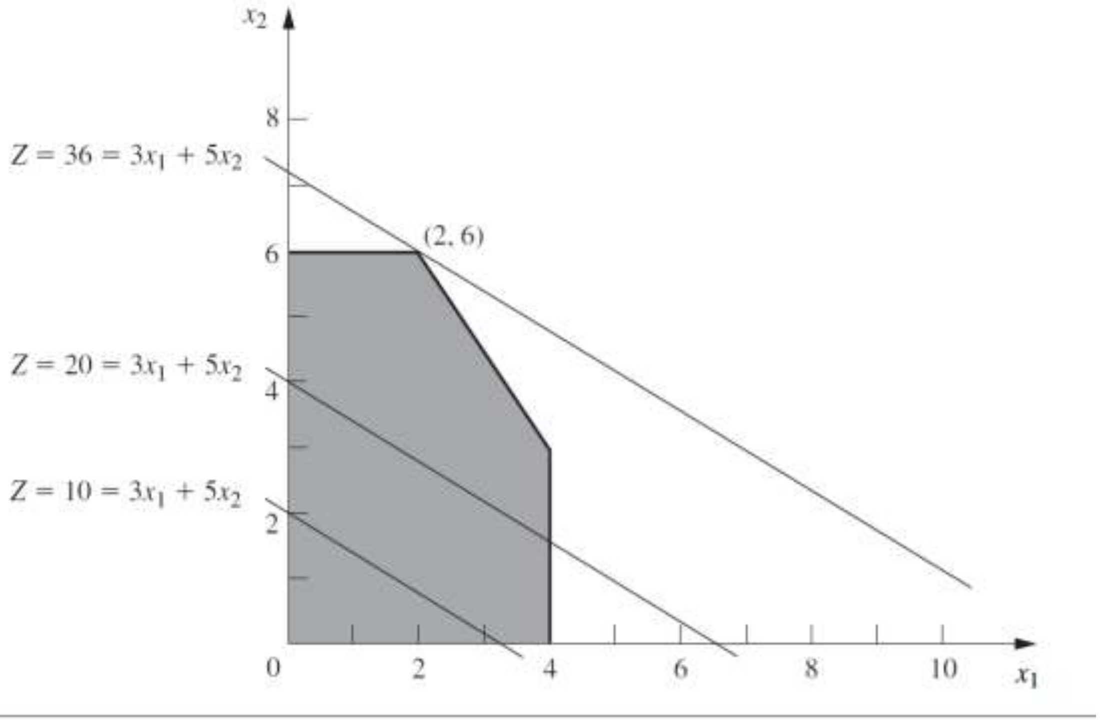
upf. BARCELONA SCHOOL OF MANAGEMENT



Bar mistakes…

How to handle the objective function to be maximized $Z = 3x_1 + 5x_2$?

Giving arbitrary values to Z results in several straight lines, all parallel to one another

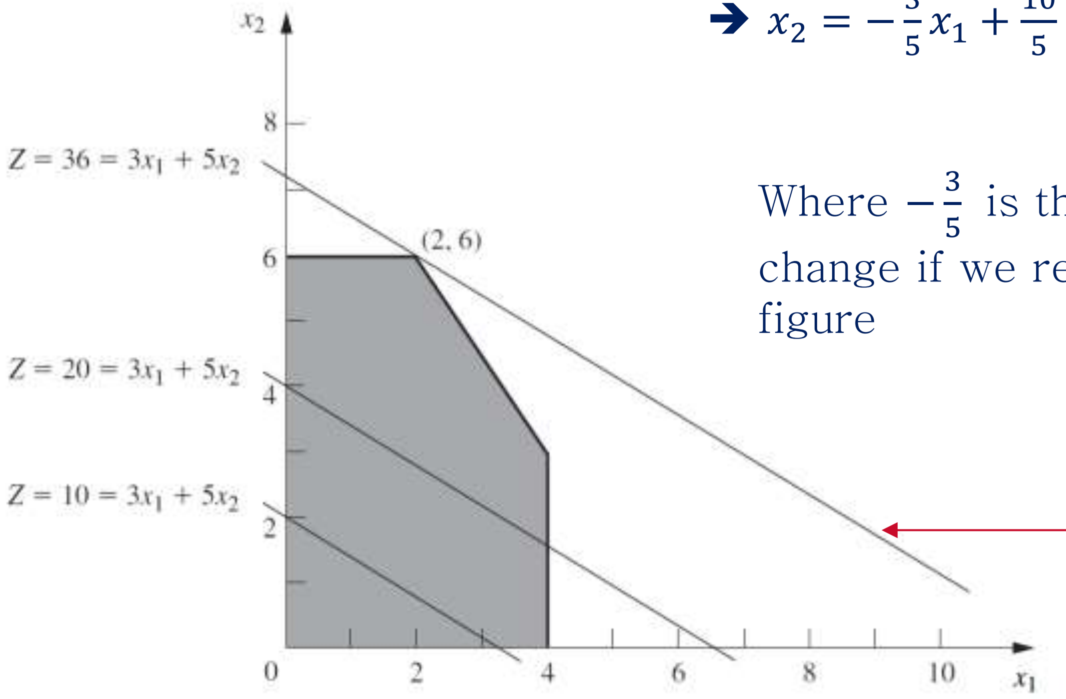


Giving arbitrary values to Z results in several straight lines, all parallel to one another

This is because the slope of the line is constant, e.g. if

$$3x_1 + 5x_2 = 10$$

$$\rightarrow x_2 = -\frac{3}{5}x_1 + \frac{10}{5}$$



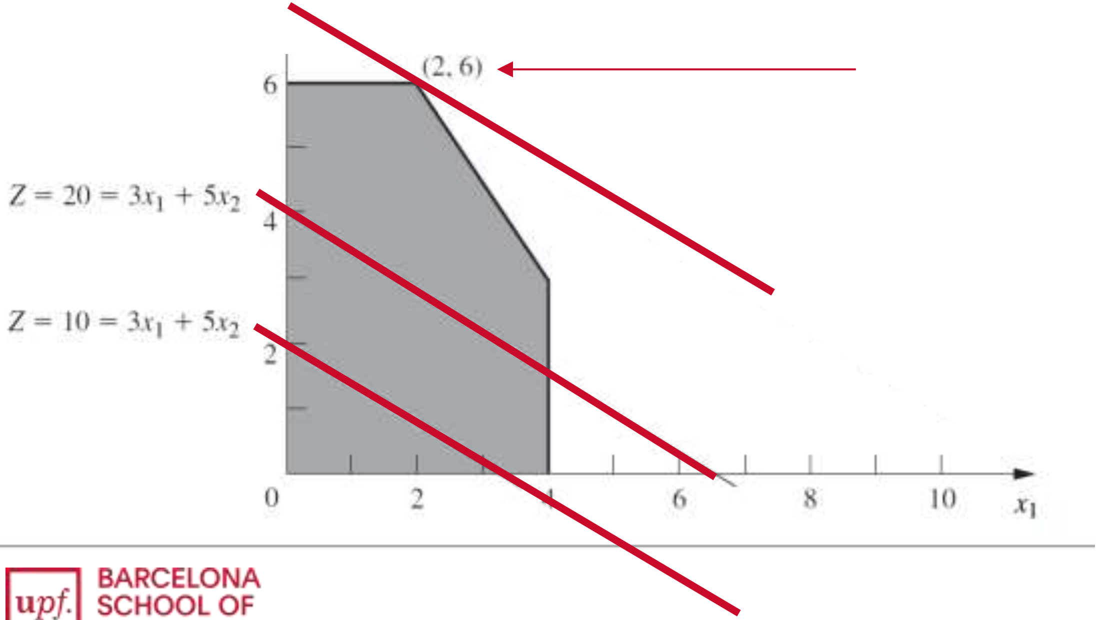
Where $-\frac{3}{5}$ is the slope of line; it does not change if we replace 10 with 20 or 36 as in the figure

In case this line has not been drawn already, how could you find it?

First we need to note that Z grows from bottom up.

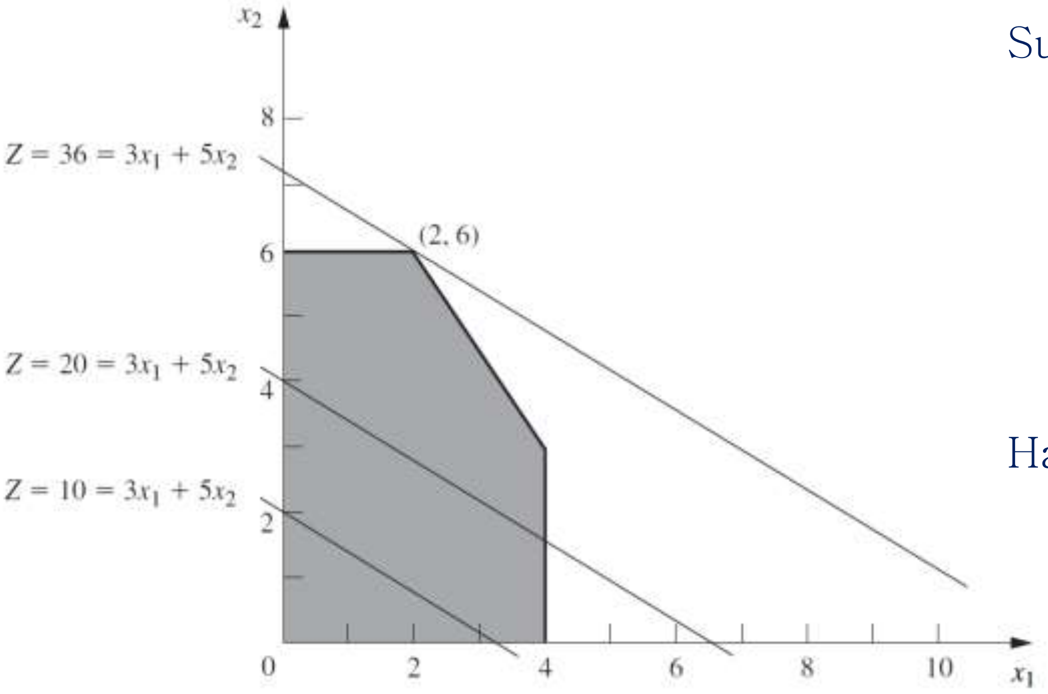
Then to note that if we must respect $Z = 3x_1 + 5x_2$ the best point (x_1, x_2)=(2,6)

Then we just plug (x_1, x_2)=(2,6) into $Z = 3x_1 + 5x_2$ to get $Z = 3 * 2 + 5 * 6 = 36$



BARCELONA SCHOOL OF MANAGEMENT

The value of (x_1, x_2) that maximizes $3x_1 + 5x_2$ is (2, 6).



The problem

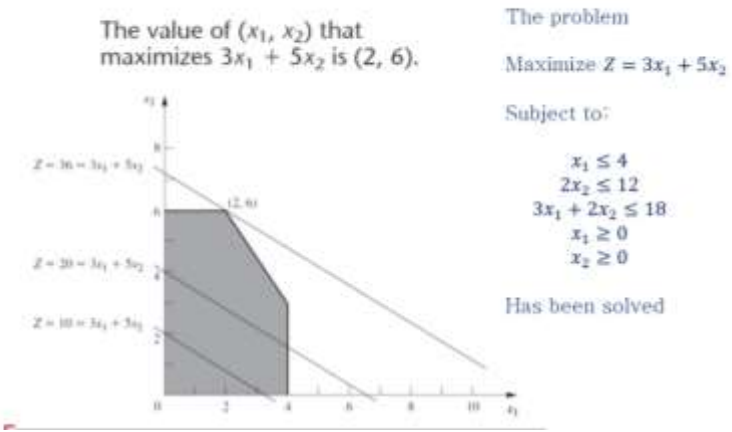
Maximize $Z = 3x_1 + 5x_2$

Subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Has been solved

upf. SCHOOL OF MANAGEMENT

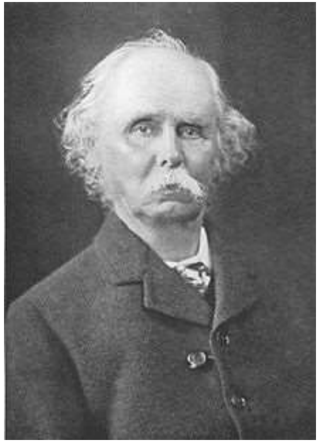


In plain English: we identified the feasible region (grey); then studied which way the optimization curve moved increasing Z (up); finally found the best point where this curve 'touched' the feasible region (2,6)

Which economist was the first to stress the importance express math in plain English?

(1) Use mathematics as a shorthand language, rather than as an engine of inquiry.
(2) Keep to them till you have done.
(3) Translate into English.
(4) Then illustrate by examples that are important in real life.
(5) Burn the mathematics.
(6) If you can't succeed in (4), burn (3). This last I [Marshall] did often.

Alfred Marshall, Memorials of Alfred Marshall, ed. A.C. Pigou (London: Macmillan, 1925), 427



Alfred Marshall
1842–1924

upf. BARCELONA SCHOOL OF MANAGEMENT

It is instructive to see what happens if

Maximize $Z = 3x_1 + 5x_2$

is replaced by

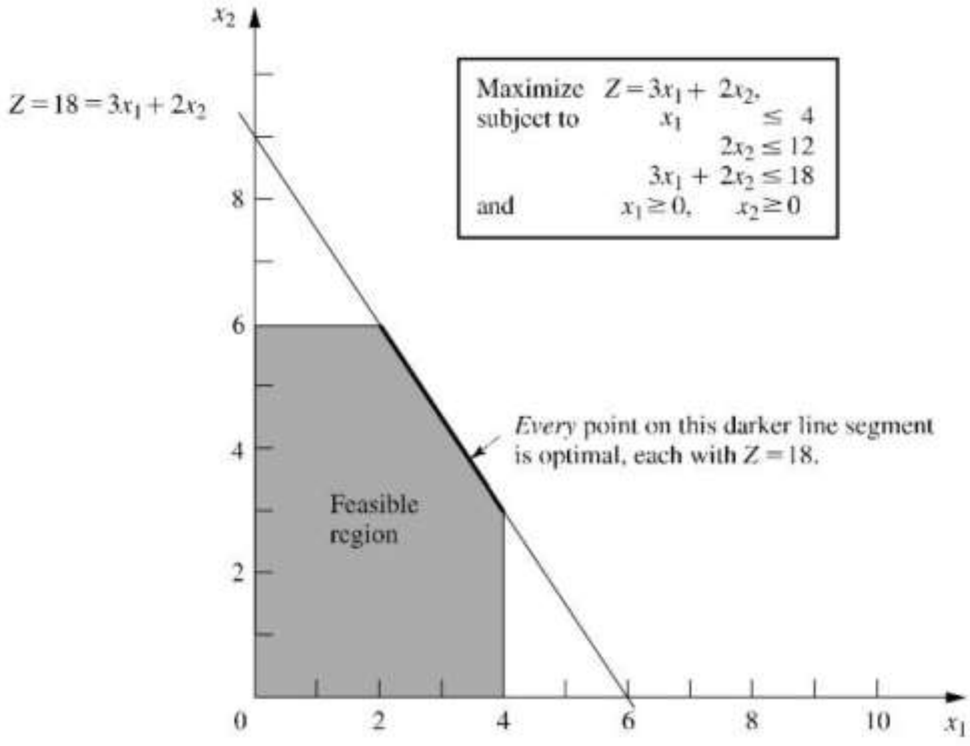
Maximize $Z = 3x_1 + 2x_2$

Source: The Simpson, 20th Television Animation (The Walt Disney Company)

Still subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Paper, pencil ad ruler: please try this out on a Cartesian diagram



It is instructive to see what happens if

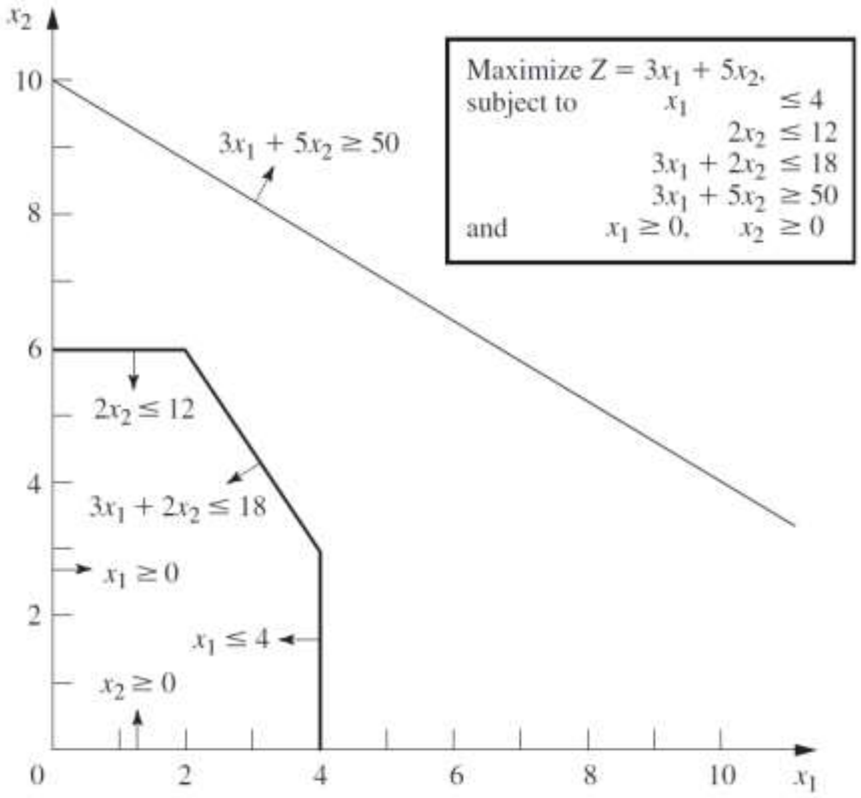
Maximize $Z = 3x_1 + 5x_2$

is replaced by

Maximize $Z = 3x_1 + 2x_2$

Still subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$



Not all problems have a solution

Maximize $Z = 3x_1 + 5x_2$

Subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$3x_1 + 5x_2 \geq 50$$ added a constrain
$$x_1 \geq 0$$
$$x_2 \geq 0$$

upf. BARCELONA SCHOOL OF MANAGEMENT

A Standard Form of the Model:

$$\text{Maximize} \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

Subject to:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m, \end{aligned}$$

And to:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \ldots, \quad x_n \geq 0.$$

Z = value of overall measure of performance

x_j = decision variables, level of activity j for $j = 1,2,..n$

a_j^i = amount of resource i consumed by each unit of activity j

b_i amount of resource i, for $i = 1,2,\ldots m,$ that is available for allocation to activities

c_j increase in Z that would result from each unit increase in level of activity

BARCELONA SCHOOL OF MANAGEMENT

A Standard Form of the Model:

Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$ Objective function

Subject to:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m,
\end{aligned}
$$

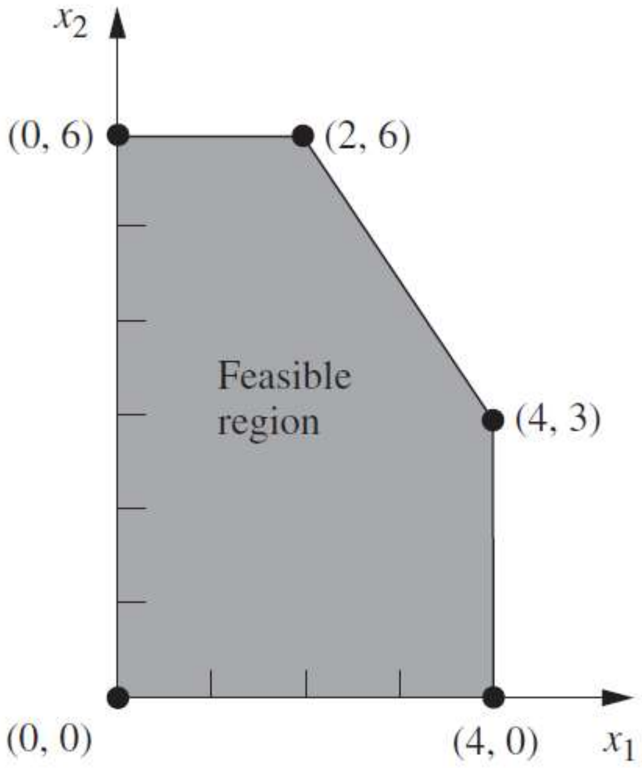
Functional constraints

And to:

$$x_1 \ge 0, \quad x_2 \ge 0, \quad \ldots, \quad x_n \ge 0.$$

Nonegativity constraints

upf. BARCELONA SCHOOL OF MANAGEMENT

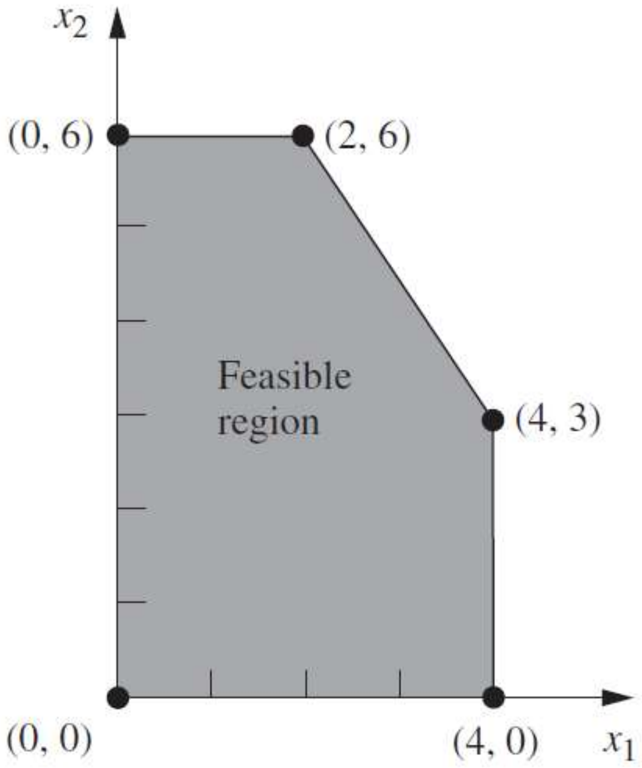


The fact that our solution in on a corner point of the feasible region is key to the theory of linear programming

Definition: A corner-point feasible (CPF) solution is a solution that lies at a corner of the feasible region

There are five CPF's in the figure

BARCELONA SCHOOL OF MANAGEMENT



Definition: A corner-point feasible (CPF) solution is a solution that lies at a corner of the feasible region

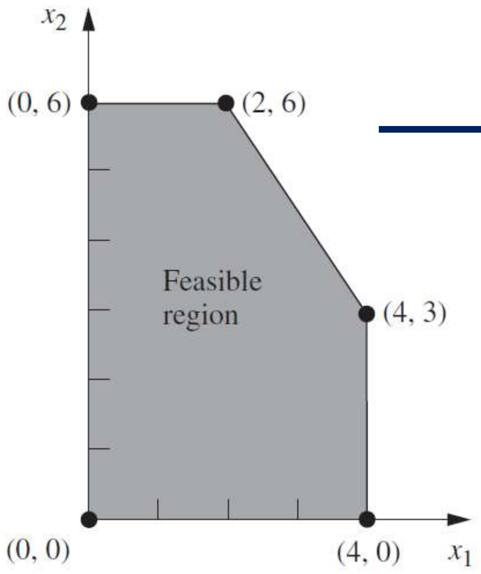
There are five CPF’s in the figure

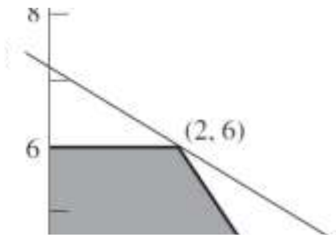
Any linear programming problem with feasible solutions and a bounded feasible region must possess CPF solutions and at least one optimal solution

Furthermore, the best CPF solution must be an optimal solution

Thus, if a problem has exactly one optimal solution, it must be a CPF solution. If the problem has multiple optimal solutions, at least two must be CPF solutions

BARCELONA SCHOOL OF MANAGEMENT

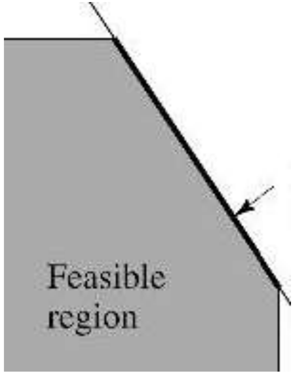




Z touches one CPF

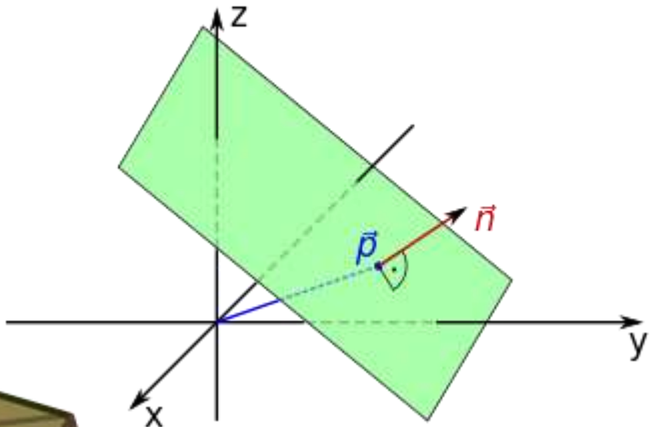
A “hand waiving” explanation:

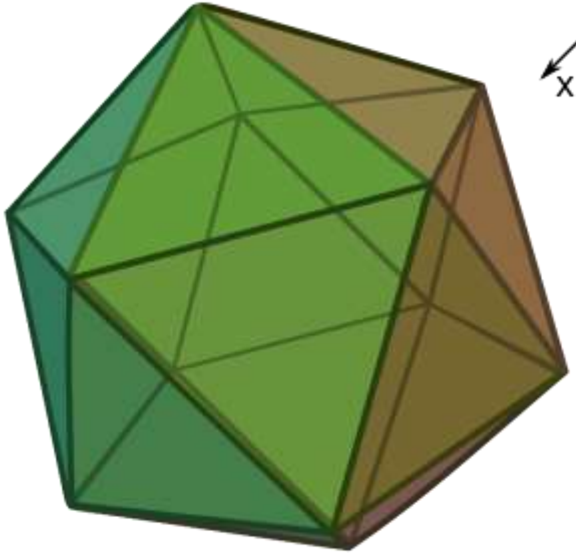
In two dimensions the corner points generated by the constraints and the straight line representing the objective function touch one another



Z touches two CPF’s

upf. BARCELONA SCHOOL OF MANAGEMENT





A "hand waiving" explanation:

In n dimensions the feasible region is a hyper polyhedron while the objective function is a plane; when it touches the polyhedron it will be in on a CPF (a corner) – or if there are more solutions, it will touch at least two CPF's (an edge or a plane)

Source (both images): Wikipedia Commons

BARCELONA SCHOOL OF MANAGEMENT

For next lesson, bring laptop with MS Excel and its SOLVER installed

How to instal and open MS EXCEL SOLVER?

In MAC

https://www.youtube.com/watch?v=ge4FMyZEUF0

In Windows

https://www.youtube.com/watch?v=W6tIS4JZ5J0

upf. BARCELONA SCHOOL OF MANAGEMENT

6.

Assumptions

Assumption made in linear programming. Hillier 2014, chapter 3.

BARCELONA SCHOOL OF MANAGEMENT

Assumptions of linear programming

Proportionality: The contribution of each activity to the value of the objective function Z is proportional to the level of the activity x_j increase in Z that , as represented by the $c_j x_j$ term in the objective function

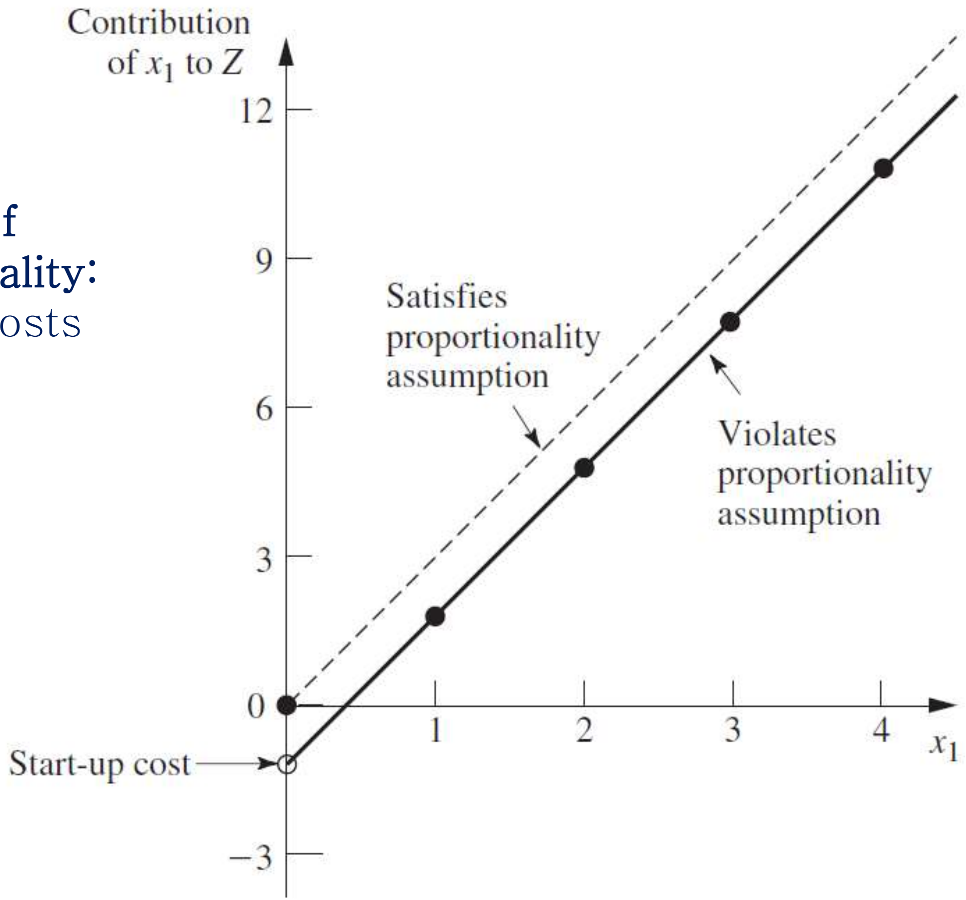
BARCELONA SCHOOL OF MANAGEMENT

Assumptions of linear programming

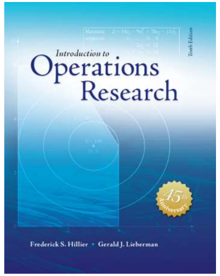
Proportionality: The contribution of each activity to the value of the objective function Z is proportional to the level of the activity x_j increase in the objective function Z, as represented by the $c_j x_j$ terms

$$\text{Maximize} \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

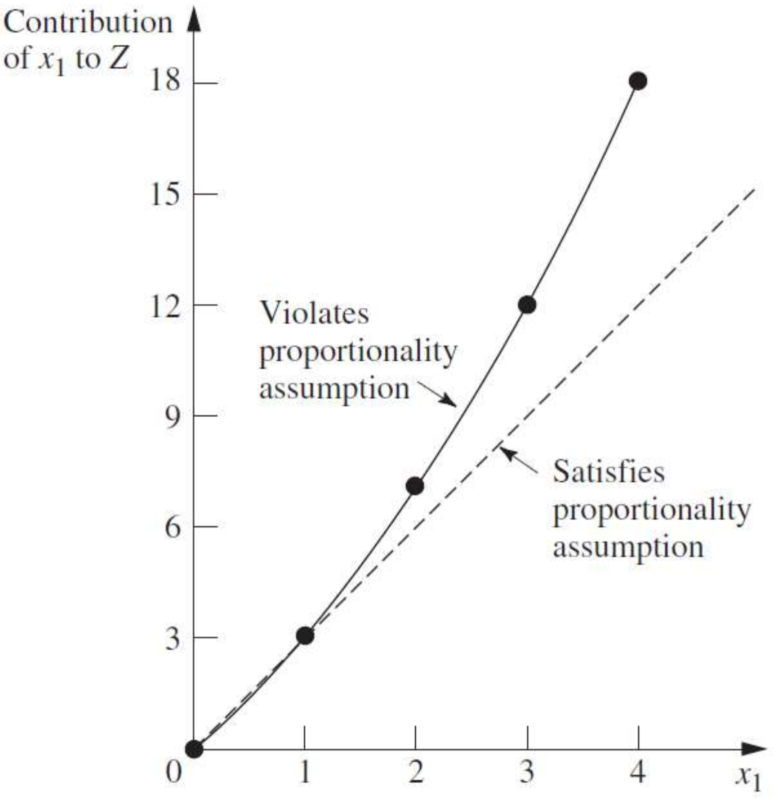
Violation of Proportionality: Start-up costs



The solid curve violates the proportionality assumption because of the start-up cost



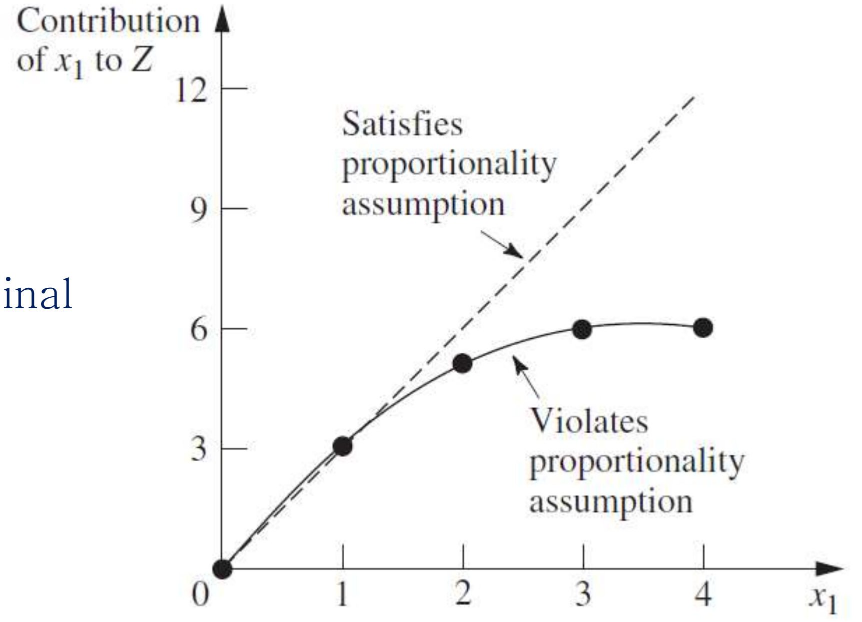
Violation of Proportionality: Increasing marginal returns (Mercedes, iPhones)



The solid curve violates the proportionality assumption because its slope (the marginal return from product 1) keeps increasing as x_1 is increased

When can this happen?

Violation of Proportionality:
Diminishing marginal returns (bananas, copper)

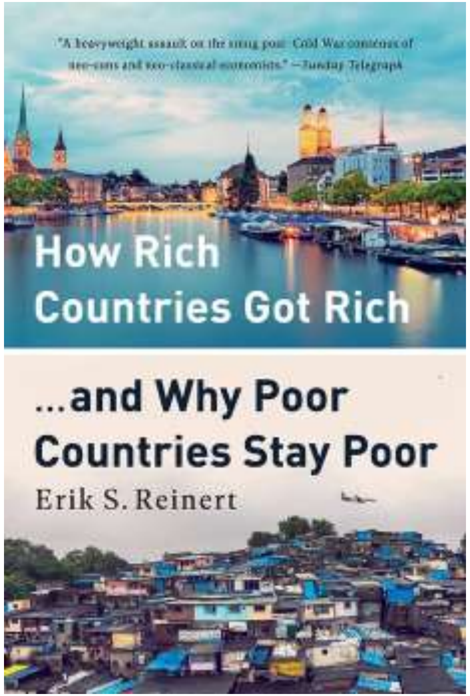


The solid curve violates the proportionality assumption because its slope (the marginal return from product 1) keeps decreasing as x_1 is increased

When can this happen?

upf. BARCELONA SCHOOL OF MANAGEMENT

Diminishing (bananas, copper) versus increasing (Mercedes, iPhones) marginal returns can make the difference between rich and poor countries



Erik S. Reinert

BARCELONA SCHOOL OF MANAGEMENT

Assumptions of linear programming

Additivity: Every function in a linear programming model (whether the objective function or the function on the left-hand side of a functional constraint) is the sum of the individual contributions of the respective activities

Additive? →

Maximize $Z = 3x_1 + 5x_2 + x_1x_2$

Subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

upf. BARCELONA SCHOOL OF MANAGEMENT

Assumptions of linear programming

Divisibility: Decision variables in a linear programming model are allowed to have any values, including noninteger values, that satisfy the functional and nonnegativity constraints. Thus, these variables are not restricted to just integer values. Since each decision variable represents the level of some activity, it is being assumed that the activities can be run at fractional levels

When a decision variable **must** be an integer, it becomes a case of integer programming

Knapsack problem algorithm

Source: https://victoria.dev/blog/knapsack-problem-algorithms-for-my-real-life-carry-on-knapsack/

Can this be formulated as a linear programming problem?

Yes, items with different utility to be packed without exceeding a given total weight

Does divisibility apply?

Not with these items

With other items?

Assumptions of linear programming

Certainty: The value assigned to the parameters (the a_j^i's, b_i's, and c_j's) of a linear programming model are assumed to be known constants

> "it is usually important to conduct sensitivity analysis after a solution is found that is optimal under the assumed parameter values" (Hillier, p. 43)
>
> "For a mathematical model with specified values for all its parameters, the model's sensitive parameters are the parameters whose value cannot be changed without changing the optimal solution" (Hillier, p. 17)

In practice what is checked in linear programming's sensitivity analysis is which parameter – when moved – can change the optimal solutions, and this is done moving each parameter at a time

This approach is consistent with the optimization logic but becomes fragile when for example (a) more parameters are uncertain or (b) the system has non linearities / non additivities ….

upf. BARCELONA SCHOOL OF MANAGEMENT

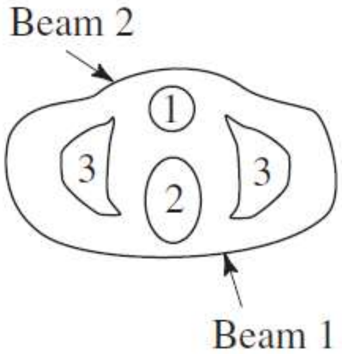
7.

More examples

More examples of linear programming. Hillier 2014, chapter 3.

BARCELONA SCHOOL OF MANAGEMENT

More cases: (1) Design of Radiation Therapy for patient Mary



1. Bladder and tumor
2. Rectum, coccyx, etc.
3. Femur, part of pelvis, etc.

FIGURE 3.11
Cross section of Mary's tumor (viewed from above), nearby critical tissues, and the radiation beams being used.

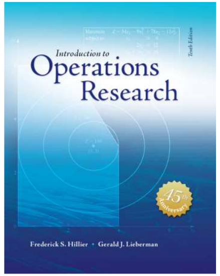
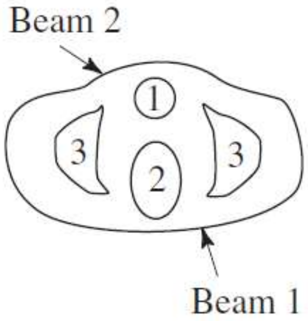


TABLE 3.7 Data for the design of Mary's radiation therapy

Area	Fraction of Entry Dose Absorbed by Area (Average)		Restriction on Total Average Dosage, Kilorads
	Beam 1	Beam 2	
Healthy anatomy	0.4	0.5	Minimize
Critical tissues	0.3	0.1	$\leq$ 2.7
Tumor region	0.5	0.5	$=$ 6
Center of tumor	0.6	0.4	$\geq$ 6



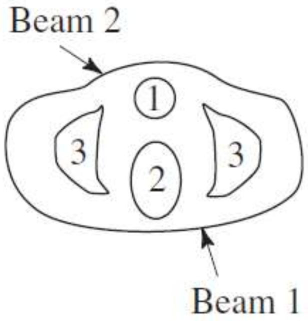
1. Bladder and tumor
2. Rectum, coccyx, etc.
3. Femur, part of pelvis, etc.

The data consist of how much radiation will be received by each of the four areas (tumour and non-tumour) from each of the two beams

upf. BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.7 Data for the design of Mary's radiation therapy

	Fraction of Entry Dose Absorbed by Area (Average)		
Area	**Beam 1**	**Beam 2**	**Restriction on Total Average Dosage, Kilorads**
Healthy anatomy	0.4	0.5	Minimize
Critical tissues	0.3	0.1	≤ 2.7
Tumor region	0.5	0.5	= 6
Center of tumor	0.6	0.4	≥ 6



1. Bladder and tumor
2. Rectum, coccyx, etc.
3. Femur, part of pelvis, etc.

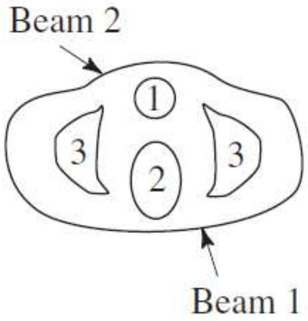
"For example, if the dose level at the entry point for beam 1 is 1 kilorad, then an average of 0.4 kilorad will be absorbed by the entire healthy anatomy in the two-dimensional plane, an average of 0.3 kilorad will be absorbed by nearby critical tissues, an average of 0.5 kilorad will be absorbed by the various parts of the tumour, and 0.6 kilorad will be absorbed by the centre of the tumour."

BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.7 Data for the design of Mary's radiation therapy

=per Kilorad

Area	Fraction of Entry Dose Absorbed by Area (Average)		Restriction on Total Average Dosage, Kilorads
	Beam 1	Beam 2	
Healthy anatomy	0.4	0.5	Minimize
Critical tissues	0.3	0.1	≤ 2.7
Tumor region	0.5	0.5	= 6
Center of tumor	0.6	0.4	≥ 6



1. Bladder and tumor
2. Rectum, coccyx, etc.
3. Femur, part of pelvis, etc.

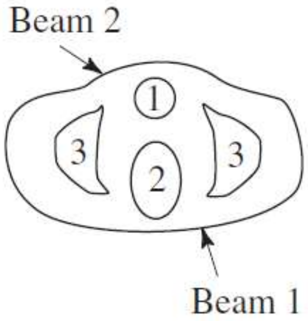
Decision variables?

a) Time of exposure beams 1 and 2?
b) Fraction of entry dose from beams 1 and 2
c) Dosages from beams 1 and 2 (Kilorads)

upf. BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.7 Data for the design of Mary's radiation therapy

Area	Fraction of Entry Dose Absorbed by Area (Average)		Restriction on Total Average Dosage, Kilorads
	Beam 1	Beam 2	
Healthy anatomy	0.4	0.5	Minimize
Critical tissues	0.3	0.1	≤ 2.7
Tumor region	0.5	0.5	= 6
Center of tumor	0.6	0.4	≥ 6



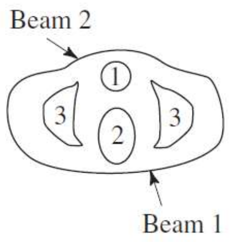
1. Bladder and tumor
2. Rectum, coccyx, etc.
3. Femur, part of pelvis, etc.

d) Dosages from beams 1 and 2 (Kilorads)

BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.7 Data for the design of Mary's radiation therapy

Area	Fraction of Entry Dose Absorbed by Area (Average)		Restriction on Total Average Dosage, Kilorads
	Beam 1	Beam 2	
Healthy anatomy	0.4	0.5	Minimize
Critical tissues	0.3	0.1	≤ 2.7
Tumor region	0.5	0.5	= 6
Center of tumor	0.6	0.4	≥ 6



1. Bladder and tumor
2. Rectum, coccyx, etc.
3. Femur, part of pelvis, etc.

Minize $\mathbf{Z} = 0.4x_1 + 0.5x_2$ Subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$0.6x_1 + 0.4x_2 \geq 6$$

These are the … Structural constraints

And

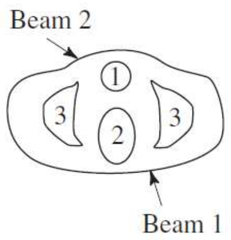
$$x_1 \geq 0$$
$$x_1 \geq 0$$

These are the … Nonegativity constraints

BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.7 Data for the design of Mary's radiation therapy

Area	Fraction of Entry Dose Absorbed by Area (Average)		Restriction on Total Average Dosage, Kilorads
	Beam 1	Beam 2	
Healthy anatomy	0.4	0.5	Minimize
Critical tissues	0.3	0.1	$\leq$ 2.7
Tumor region	0.5	0.5	= 6
Center of tumor	0.6	0.4	$\geq$ 6



1. Bladder and tumor
2. Rectum, coccyx, etc.
3. Femur, part of pelvis, etc.

Minize $Z = 0.4x_1 + 0.5x_2$ Subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$0.6x_1 + 0.4x_2 \geq 6$$

And

$$x_1 \geq 0$$
$$x_1 \geq 0$$

What is new in this case?

upf. BARCELONA SCHOOL OF MANAGEMENT

Minize $Z = 0.4x_1 + 0.5x_2$

Subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$0.6x_1 + 0.4x_2 \geq 6$$

And

$$x_1 \geq 0$$
$$x_1 \geq 0$$

Time for work on the Cartesian plane

Hint:

1) start by drawing the straight lines

$$0.3x_1 + 0.1x_2 = 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$0.6x_1 + 0.4x_2 = 6$$

2) identify the critical region
3) Compute Z at the extremes of the critical region – for this you must find the intersections of the various lines

BARCELONA SCHOOL OF MANAGEMENT

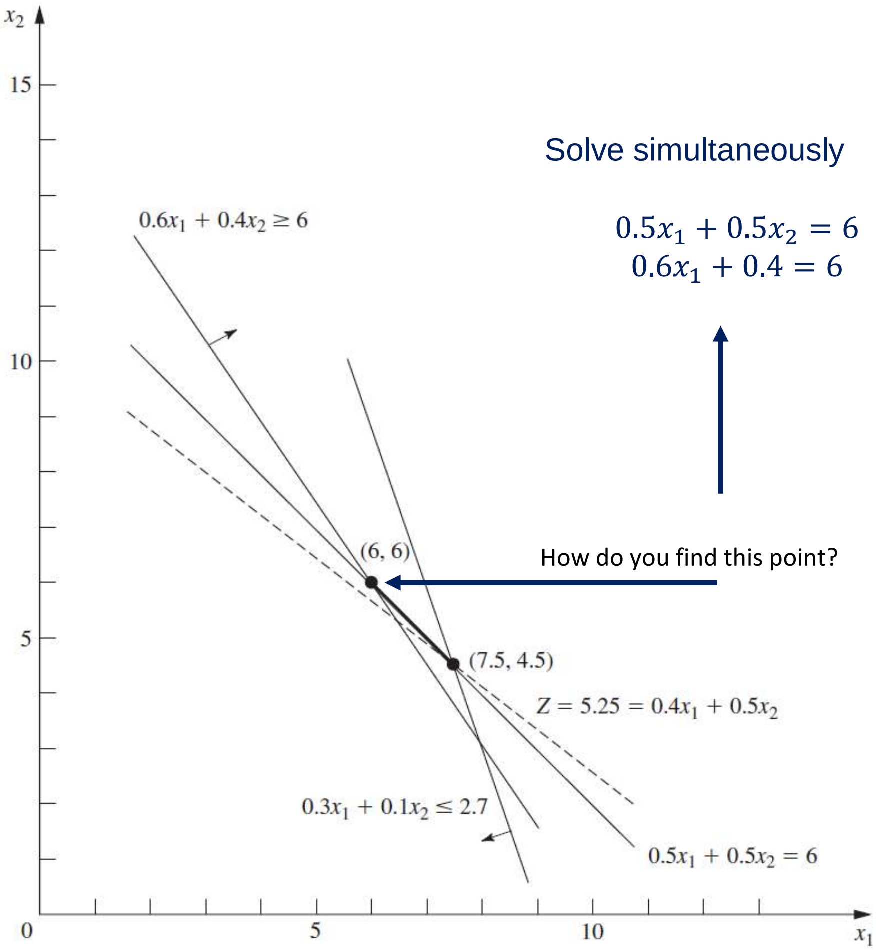
Minize $Z = 0.4x_1 + 0.5x_2$

Subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$0.6x_1 + 0.4x_2 \geq 6$$

And

$$x_1 \geq 0$$
$$x_1 \geq 0$$



BARCELONA SCHOOL OF MANAGEMENT

To solve simultaneously

$0.5x_1 + 0.5x_2 = 6$
$0.6x_1 + 0.4x_2 = 6$

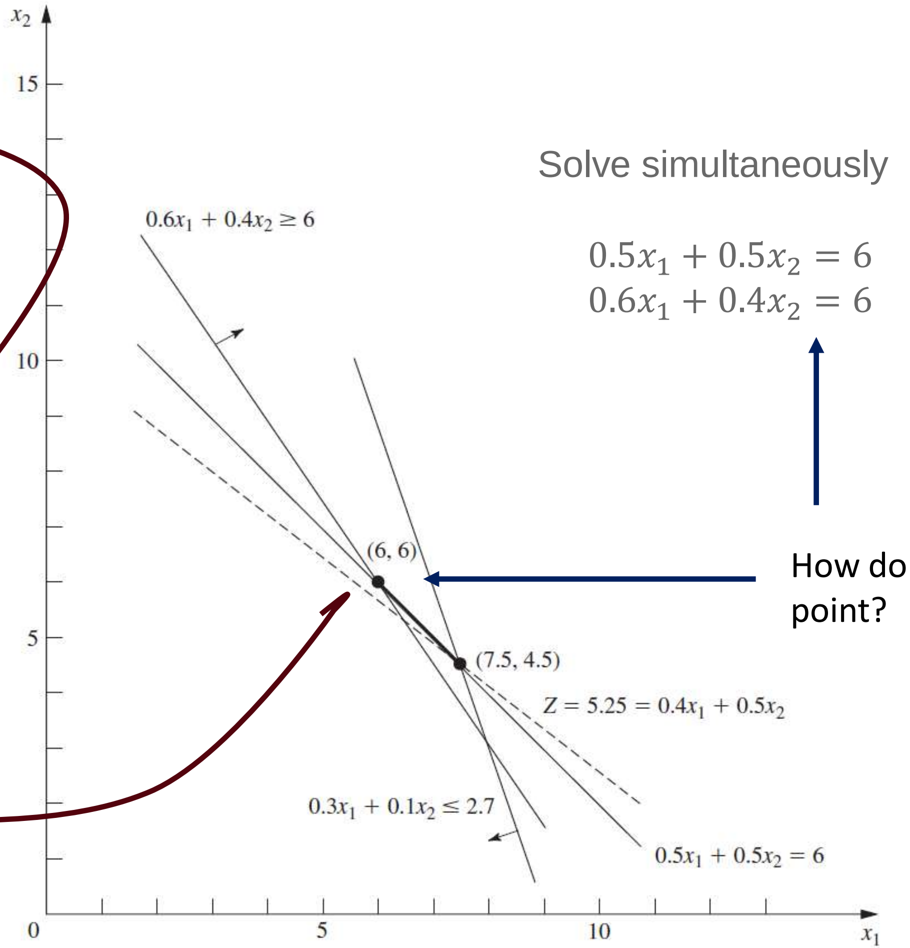
Derive x_1 from the first equation

$x_1 = \left(\frac{-0.5x_2+6}{0.5}\right) = -x_2 + 12$

Plug this into the second equation

$$0.6(12 - x_2) + 0.4x_2 = 6$$
$$7.2 - 0.2x_2 = 6$$
$$x_2 = 6$$

Plugging this back in either the first or the second equation gives $x_1 = 6$



Solve simultaneously

$0.5x_1 + 0.5x_2 = 6$
$0.6x_1 + 0.4x_2 = 6$

How do you find this point?

BARCELONA SCHOOL OF MANAGEMENT

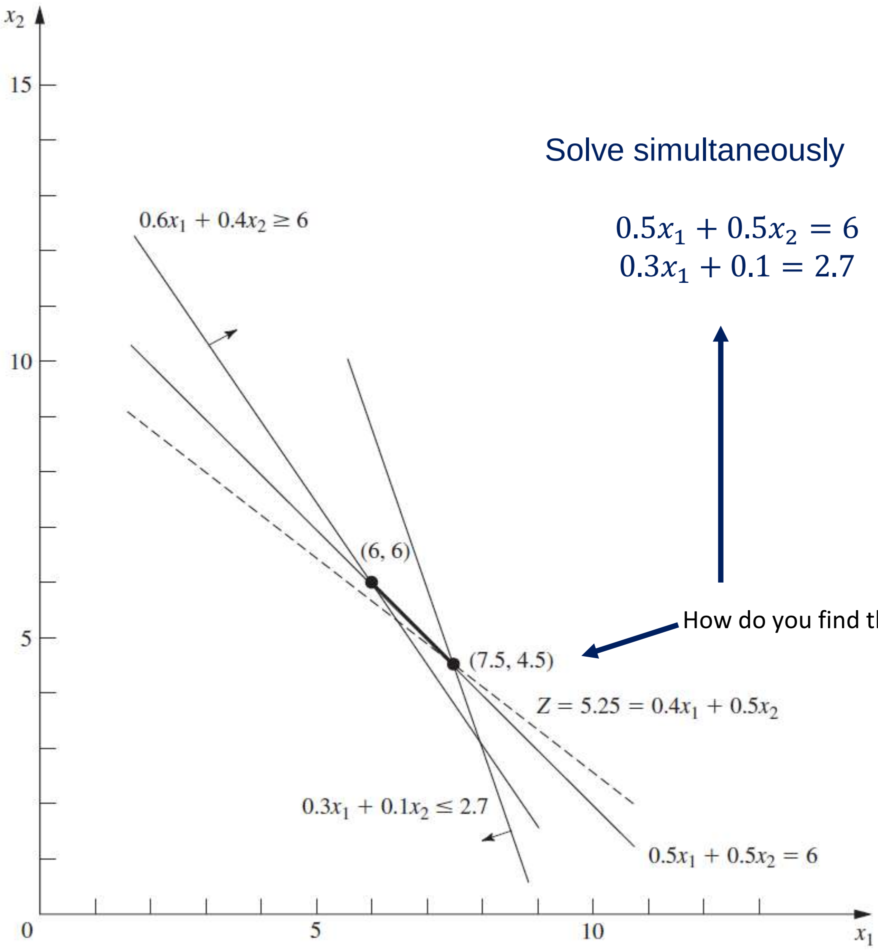
Minize $Z = 0.4x_1 + 0.5x_2$

Subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$0.6x_1 + 0.4x_2 \geq 6$$

And

$$x_1 \geq 0$$
$$x_1 \geq 0$$



Solve simultaneously

$$0.5x_1 + 0.5x_2 = 6$$
$$0.3x_1 + 0.1 = 2.7$$

How do you find this point?

BARCELONA SCHOOL OF MANAGEMENT

To solve simultaneously

$0.5x_1 + 0.5x_2 = 6$

$0.3x_1 + 0.1x_2 = 2.7$

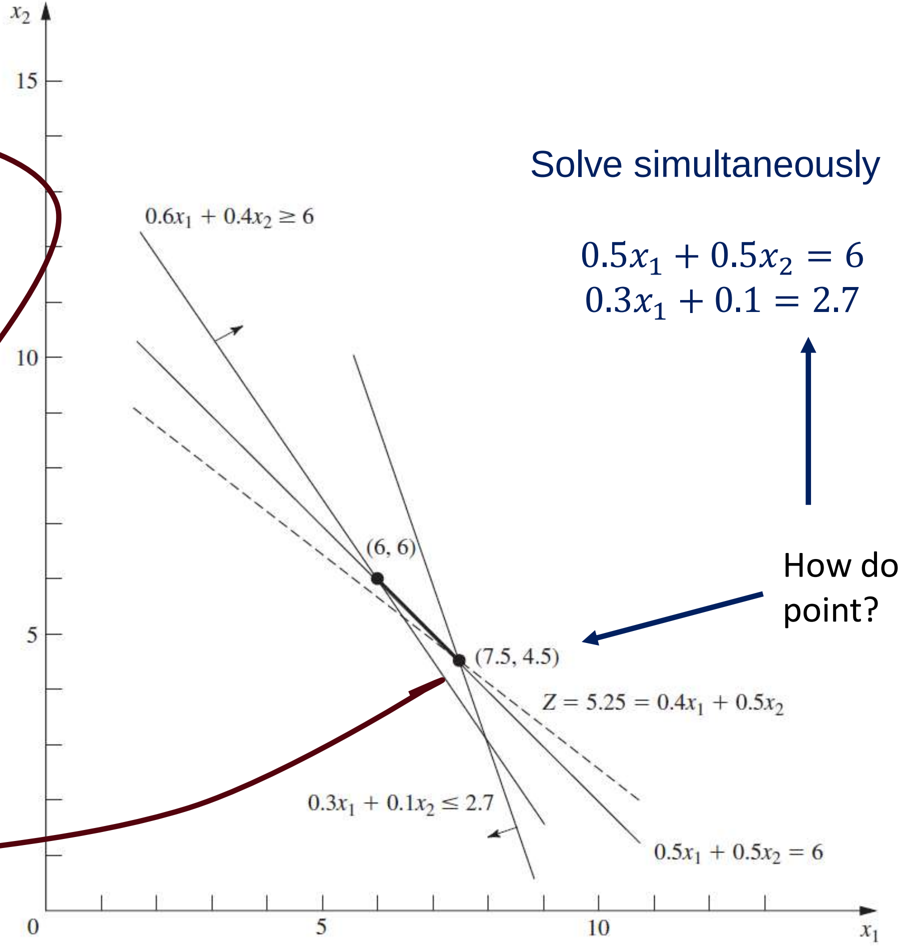
Derive x_1 from the first equation

$x_1 = \left(\frac{-0.5x_2+6}{0.5}\right) = -x_2 + 12$

Plug this into the second equation

$$0.3(12 - x_2) + 0.1x_2 = 2.7$$
$$3.6 - 0.3x_2 + 0.1x_2 = 2.7$$
$$0.2x_2 = 0.9$$
$$x_2 = 4.5$$

Plugging this back in either the first or the second equation gives $x_1 = 7.5$



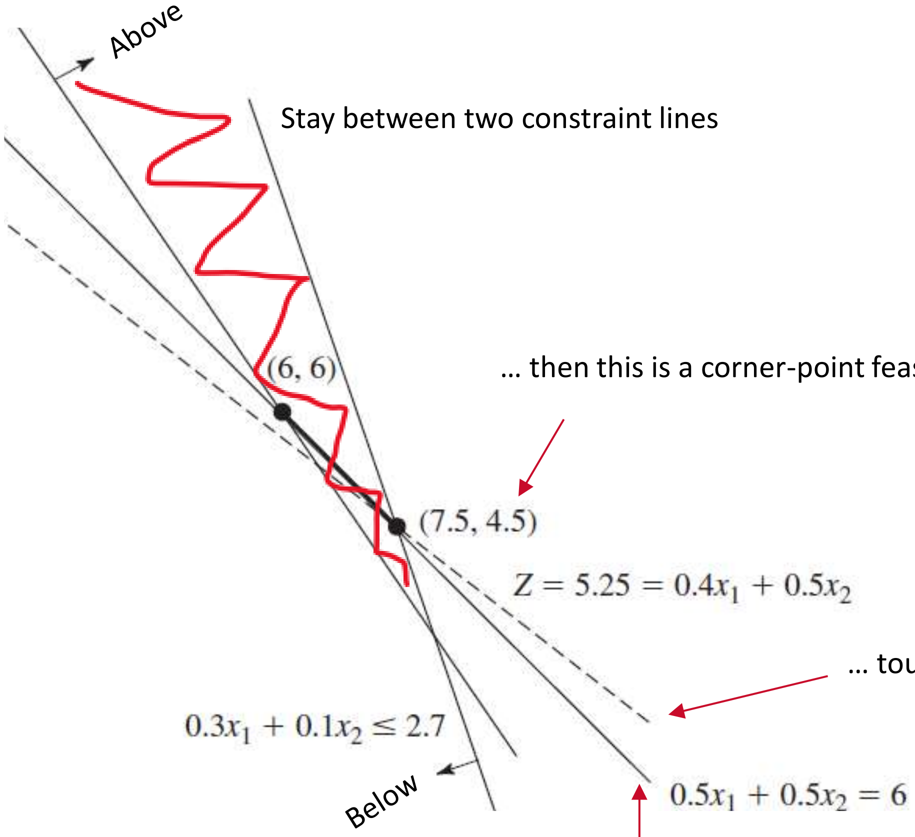
Solve simultaneously

$0.5x_1 + 0.5x_2 = 6$

$0.3x_1 + 0.1 = 2.7$

How do you find this point?

BARCELONA SCHOOL OF MANAGEMENT



Stay between two constraint lines

… then this is a corner-point feasible (CPF) solution

… touched by the optimization line

…but also stay on this line

More cases: (2) Controlling Air Pollution
A steel producing company needs to cut the emissions from one of its plans.
The desired reduction is:

TABLE 3.12 Clean air standards for the Nori & Leets Co.

Pollutant	Required Reduction in Annual Emission Rate (Million Pounds)
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

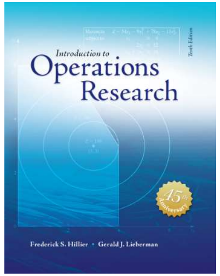


TABLE 3.12 Clean air standards for the Nori & Leets Co.

Pollutant	Required Reduction in Annual Emission Rate (Million Pounds)
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

The pollution arises from two primary sources, namely, the blast furnaces for making pig iron and the open-hearth furnaces for changing iron into steel.

Used at full power, the three methods available to reduce emissions (taller smokestacks, filters and better fuel) will yield the following reduction

TABLE 3.13 Reduction in emission rate (in millions of pounds per year) from the maximum feasible use of an abatement method for Nori & Leets Co.

	Taller Smokestacks		Filters		Better Fuels	
Pollutant	**Blast Furnaces**	**Open-Hearth Furnaces**	**Blast Furnaces**	**Open-Hearth Furnaces**	**Blast Furnaces**	**Open-Hearth Furnaces**
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

TABLE 3.12 Clean air standards for the Nori & Leets Co.

Pollutant	Required Reduction in Annual Emission Rate (Million Pounds)
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

The pollution arises from two primary sources: the blast furnaces and the open-hearth furnaces

TABLE 3.13 Reduction in emission rate (in millions of pounds per year) from the maximum feasible use of an abatement method for Nori & Leets Co.

	Taller Smokestacks		Filters		Better Fuels	
Pollutant	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

Used at full power, the three methods available to reduce emissions (taller smokestacks, filters and better fuel) will yield the following reduction

Decision variables?

TABLE 3.14 Total annual cost from the maximum feasible use of an abatement method for Nori & Leets Co. ($ millions)

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	8	10
Filters	7	6
Better fuels	11	9

And this is the associated cost, still using the methods at their fullest power

TABLE 3.12 Clean air standards for the Nori & Leets Co.

Pollutant	Required Reduction in Annual Emission Rate (Million Pounds)
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

TABLE 3.13 Reduction in emission rate (in millions of pounds per year) from the maximum feasible use of an abatement method for Nori & Leets Co.

	Taller Smokestacks		Filters		Better Fuels	
Pollutant	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

Look at the constraints, expressed as function of maximum feasible use …

Decision variables?

TABLE 3.14 Total annual cost from the maximum feasible use of an abatement method for Nori & Leets Co. ($ millions)

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	8	10
Filters	7	6
Better fuels	11	9

Look at the structure of the cost; it depends on the three methods applied to the two furnaces …

Decision variables?

So we go from this

■ **TABLE 3.14** Total annual cost from the maximum feasible use of an abatement method for Nori & Leets Co. ($ millions)

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	8	10
Filters	7	6
Better fuels	11	9

To this

■ **TABLE 3.15** Decision variables (fraction of the maximum feasible use of an abatement method) for Nori & Leets Co.

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	x_1	x_2
Filters	x_3	x_4
Better fuels	x_5	x_6

Perhaps the fraction of method $i = 1,2,3$ applied to furnace $j = 1,2$

This fraction can be expressed as a number in $(0,1)$

BARCELONA SCHOOL OF MANAGEMENT

Putting the two tables together

■ **TABLE 3.14** Total annual cost from the maximum feasible use of an abatement method for Nori & Leets Co. ($ millions)

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	8	10
Filters	7	6
Better fuels	11	9

■ **TABLE 3.15** Decision variables (fraction of the maximum feasible use of an abatement method) for Nori & Leets Co.

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	x_1	x_2
Filters	x_3	x_4
Better fuels	x_5	x_6

We can write

Minimize $8x_1 + 10x_2 + 7x_3 + 6x_4 + 11x_5 + 9x_6$

upf. BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.15 Decision variables (fraction of the maximum feasible use of an abatement method) for Nori & Leets Co.

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	x_1	x_2
Filters	x_3	x_4
Better fuels	x_5	x_6

TABLE 3.13 Reduction in emission rate (in millions of pounds per year) from the maximum feasible use of an abatement method for Nori & Leets Co.

	Taller Smokestacks		Filters		Better Fuels	
Pollutant	**Blast Furnaces**	**Open-Hearth Furnaces**	**Blast Furnaces**	**Open-Hearth Furnaces**	**Blast Furnaces**	**Open-Hearth Furnaces**
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

Now we have to put together these tables

We can write for particulate

$$12x_1 + 9x_2 + 25x_3 + 20x_4 + 17x_5 + 13x_6 \geq 60$$

TABLE 3.12 Clean air standards for the Nori & Leets Co.

Pollutant	Required Reduction in Annual Emission Rate (Million Pounds)
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.12 Clean air standards for the Nori & Leets Co.

Pollutant	Required Reduction in Annual Emission Rate (Million Pounds)
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

TABLE 3.13 Reduction in emission rate (in millions of pounds per year) from the maximum feasible use of an abatement method for Nori & Leets Co.

	Taller Smokestacks		Filters		Better Fuels	
Pollutant	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

The same for the other pollutants

To write:

Particulate ➔ $12x_1 + 9x_2 + 25x_3 + 20x_4 + 17x_5 + 13x_6 \geq 60$

Sulphur oxides ➔ $35x_1 + 42x_2 + 18x_3 + 31x_4 + 56x_5 + 49x_6 \geq 150$

Hydrocarbons ➔ $37x_1 + 53x_2 + 28x_3 + 24x_4 + 29x_5 + 20x_6 \geq 125$

Are we done?

upf. BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.12 Clean air standards for the Nori & Leets Co.

Pollutant	Required Reduction in Annual Emission Rate (Million Pounds)
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

TABLE 3.13 Reduction in emission rate (in millions of pounds per year) from the maximum feasible use of an abatement method for Nori & Leets Co.

	Taller Smokestacks		Filters		Better Fuels	
Pollutant	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

To write:

Particulate ➔ $12x_1 + 9x_2 + 25x_3 + 20x_4 + 17x_5 + 13x_6 \geq 60$

Sulphur oxides ➔ $35x_1 + 42x_2 + 18x_3 + 31x_4 + 56x_5 + 49x_6 \geq 150$

Hydrocarbons ➔ $37x_1 + 53x_2 + 28x_3 + 24x_4 + 29x_5 + 20x_6 \geq 125$

Nonnegativity constraints

$x_j \geq 0$ for $j = 1,2,\ldots 6$

Are we done?

$x_j \leq 1$ for $j = 1,2,\ldots 6$

BARCELONA SCHOOL OF MANAGEMENT

■ **TABLE 3.12** Clean air standards for the Nori & Leets Co.

Pollutant	Required Reduction in Annual Emission Rate (Million Pounds)
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

■ **TABLE 3.13** Reduction in emission rate (in millions of pounds per year) from the maximum feasible use of an abatement method for Nori & Leets Co.

	Taller Smokestacks		Filters		Better Fuels	
Pollutant	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces	Blast Furnaces	Open-Hearth Furnaces
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

■ **TABLE 3.14** Total annual cost from the maximum feasible use of an abatement method for Nori & Leets Co. ($ millions)

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	8	10
Filters	7	6
Better fuels	11	9

■ **TABLE 3.15** Decision variables (fraction of the maximum feasible use of an abatement method) for Nori & Leets Co.

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	x_1	x_2
Filters	x_3	x_4
Better fuels	x_5	x_6

Solved with the method of simplex (not shown here) gives the following solution:

$(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0.623, 0.343, 1, 0.048, 1)$

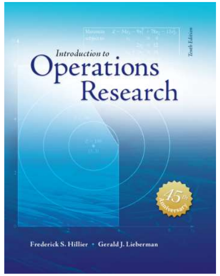
with Z=32.16

More cases: (3) Scheduling

An air company needs to allocate staff to different shifts as to cover flights while minimizing costs

The shifts are

	From time	To time
Shift 1	6:00 am	2:00 pm
Shift 2	8:00 am	4:00 pm
Shift 3	noon	8:00 pm
Shift 4	4:00 pm	midnight
Shift 5	10:00 pm	6:00 am



upf. BARCELONA SCHOOL OF MANAGEMENT

The five shifts cover different time windows at a different cost

Are these numbers needed?

TABLE 3.19 Data for the Union Airways personnel scheduling problem

Time Period	Time Periods Covered — Shift 1	2	3	4	5	Minimum Number of Agents Needed
6:00 A.M. to 8:00 A.M.	✔					48
8:00 A.M. to 10:00 A.M.	✔	✔				79
10:00 A.M. to noon	✔	✔				65
Noon to 2:00 P.M.	✔	✔	✔			87
2:00 P.M. to 4:00 P.M.		✔	✔			64
4:00 P.M. to 6:00 P.M.			✔	✔		73
6:00 P.M. to 8:00 P.M.			✔	✔		82
8:00 P.M. to 10:00 P.M.				✔		43
10:00 P.M. to midnight				✔	✔	52
Midnight to 6:00 A.M.					✔	15
Daily cost per agent	$170	$160	$175	$180	$195	

Are these numbers needed?

upf. BARCELONA SCHOOL OF MANAGEMENT

What do we want to minimize?

■ **TABLE 3.19** Data for the Union Airways personnel scheduling problem

	Time Periods Covered					
	Shift					
Time Period	**1**	**2**	**3**	**4**	**5**	**Minimum Number of Agents Needed**
6:00 A.M. to 8:00 A.M.	✔					48
8:00 A.M. to 10:00 A.M.	✔	✔				79
10:00 A.M. to noon	✔	✔				65
Noon to 2:00 P.M.	✔	✔	✔			87
2:00 P.M. to 4:00 P.M.		✔	✔			64
4:00 P.M. to 6:00 P.M.			✔	✔		73
6:00 P.M. to 8:00 P.M.			✔	✔		82
8:00 P.M. to 10:00 P.M.				✔		43
10:00 P.M. to midnight				✔	✔	52
Midnight to 6:00 A.M.					✔	15
Daily cost per agent	\$170	\$160	\$175	\$180	\$195	

Cost, based on the number x_i of agents assigned to each shift $i, i = 1,..5$:

Minimize $170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5$

BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.19 Data for the Union Airways personnel scheduling problem

Time Period	Time Periods Covered — Shift 1	2	3	4	5	Minimum Number of Agents Needed
6:00 A.M. to 8:00 A.M.	✔					48
8:00 A.M. to 10:00 A.M.	✔	✔				79
10:00 A.M. to noon	✔	✔				65
Noon to 2:00 P.M.	✔	✔	✔			87
2:00 P.M. to 4:00 P.M.		✔	✔			64
4:00 P.M. to 6:00 P.M.			✔	✔		73
6:00 P.M. to 8:00 P.M.			✔	✔		82
8:00 P.M. to 10:00 P.M.				✔		43
10:00 P.M. to midnight				✔	✔	52
Midnight to 6:00 A.M.					✔	15
Daily cost per agent	$170	$160	$175	$180	$195	

Minimize $170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5$

Which is the first structural constraint?

$$x_1 \geq 48$$

Which is the second structural constraint?

$$x_1 + x_2 \geq 79$$

BARCELONA SCHOOL OF MANAGEMENT

TABLE 3.19 Data for the Union Airways personnel scheduling problem

Time Period	Time Periods Covered — Shift 1	2	3	4	5	Minimum Number of Agents Needed
6:00 A.M. to 8:00 A.M.	✔					48
8:00 A.M. to 10:00 A.M.	✔	✔				79
10:00 A.M. to noon	✔	✔				65
Noon to 2:00 P.M.	✔	✔	✔			87
2:00 P.M. to 4:00 P.M.		✔	✔			64
4:00 P.M. to 6:00 P.M.			✔	✔		73
6:00 P.M. to 8:00 P.M.			✔	✔		82
8:00 P.M. to 10:00 P.M.				✔		43
10:00 P.M. to midnight				✔	✔	52
Midnight to 6:00 A.M.					✔	15
Daily cost per agent	$170	$160	$175	$180	$195	

Minimize $170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5$

$$x_1 \geq 48$$
$$x_1 + x_2 \geq 79$$
$$x_1 + x_2 \geq 65$$
$$x_1 + x_2 + x_3 \geq 87$$
$$x_2 + x_3 \geq 64$$
$$x_3 + x_4 \geq 73$$
$$x_3 + x_4 \geq 82$$
$$x_5 \geq 43$$
$$x_5 + x_6 \geq 52$$
$$x_6 \geq 15$$

Anything weird about these structural constraints ?

Anything Missing?

$$x_i \geq 0, i = 1, \ldots 5$$

BARCELONA SCHOOL OF MANAGEMENT

■ **TABLE 3.19** Data for the Union Airways personnel scheduling problem

Time Period	Time Periods Covered: Shift 1	2	3	4	5	Minimum Number of Agents Needed
6:00 A.M. to 8:00 A.M.	✔					48
8:00 A.M. to 10:00 A.M.	✔	✔				79
10:00 A.M. to noon	✔	✔				65
Noon to 2:00 P.M.	✔	✔	✔			87
2:00 P.M. to 4:00 P.M.		✔	✔			64
4:00 P.M. to 6:00 P.M.			✔	✔		73
6:00 P.M. to 8:00 P.M.			✔	✔		82
8:00 P.M. to 10:00 P.M.				✔		43
10:00 P.M. to midnight				✔	✔	52
Midnight to 6:00 A.M.					✔	15
Daily cost per agent	$170	$160	$175	$180	$195	

The optimal solution for this model is $(x_1, x_2, x_3, x_4, x_5)$= (48, 31, 39, 43, 15). This yields Z 30,610, that is, a total daily personnel cost of $30,610.

Minimize $170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5$

$x_i \geq 0, i = 1, \ldots 5$

TABLE 3.19 Data for the Union Airways personnel scheduling problem

Time Period	Time Periods Covered: Shift 1	2	3	4	5	Minimum Number of Agents Needed
6:00 a.m. to 8:00 a.m.	✔					48
8:00 a.m. to 10:00 a.m.	✔	✔				79
10:00 a.m. to noon	✔	✔				65
Noon to 2:00 p.m.	✔	✔	✔			87
2:00 p.m. to 4:00 p.m.		✔	✔			64
4:00 p.m. to 6:00 p.m.			✔	✔		73
6:00 p.m. to 8:00 p.m.			✔	✔		82
8:00 p.m. to 10:00 p.m.				✔		43
10:00 p.m. to midnight				✔	✔	52
Midnight to 6:00 a.m.					✔	15
Daily cost per agent	$170	$160	$175	$180	$195	

Minimize $170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5$

$$x_1 \geq 48$$
$$x_1 + x_2 \geq 79$$
$$x_1 + x_2 \geq 65$$
$$x_1 + x_2 + x_3 \geq 87$$
$$x_2 + x_3 \geq 64$$
$$x_3 + x_4 \geq 73$$
$$x_3 + x_4 \geq 82$$
$$x_5 \geq 43$$
$$x_5 + x_6 \geq 52$$
$$x_6 \geq 15$$

Anything weird about these structural constraints ?

Anything Missing?

$x_i \geq 0, i = 1, \ldots 5$

upf. BARCELONA SCHOOL OF MANAGEMENT

What happened to divisibility?

8.

George Dantzig
(1914-2005)

Method of simplex

A geometric illustration of the simplex method. Hillier 2014, chapter 4.

Simplified illustration of the simplex method
(Recall from Lesson one, developed by George Dantzig ~1947)

Corner point unfeasible solution →

Corner point feasible solution →

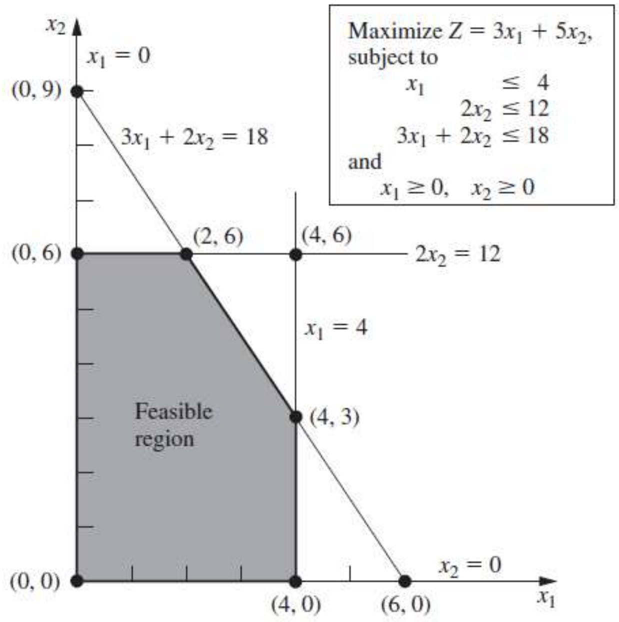
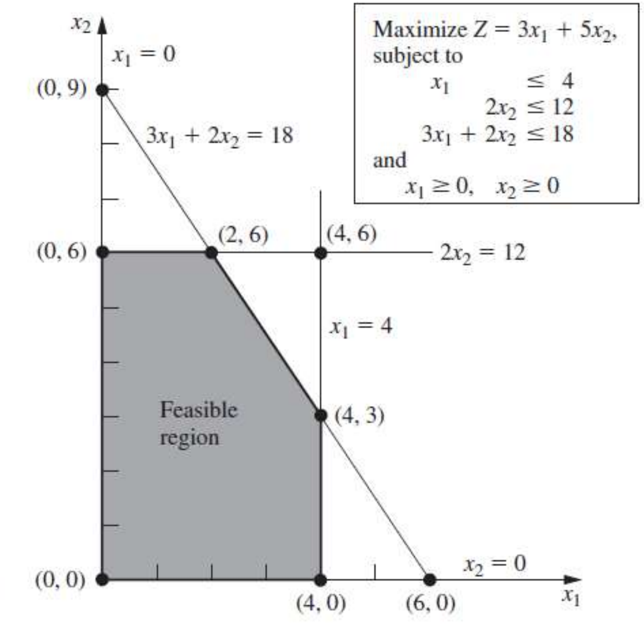


FIGURE 4.1
Constraint boundaries and corner-point solutions for the Wyndor Glass Co. problem.

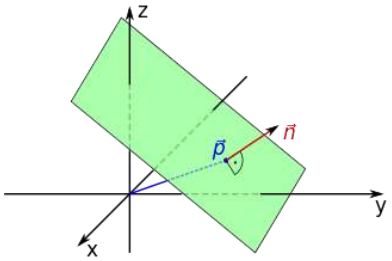
Recall the all–important concept of **Corner Point Feasible (CPF)** solution.

The problem has three unfeasible corners (which are …?) and five feasible corners (CPF) solutions (which are …?)

upf. BARCELONA SCHOOL OF MANAGEMENT



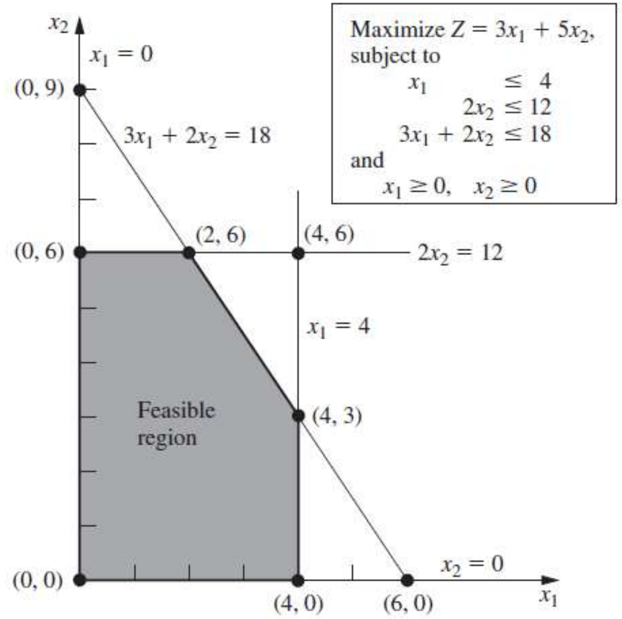
Recall that if there is only one optimal solution this must be a CPF

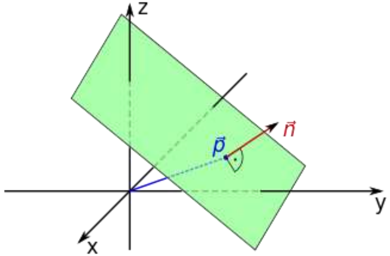


Source (both images): Wikipedia Commons

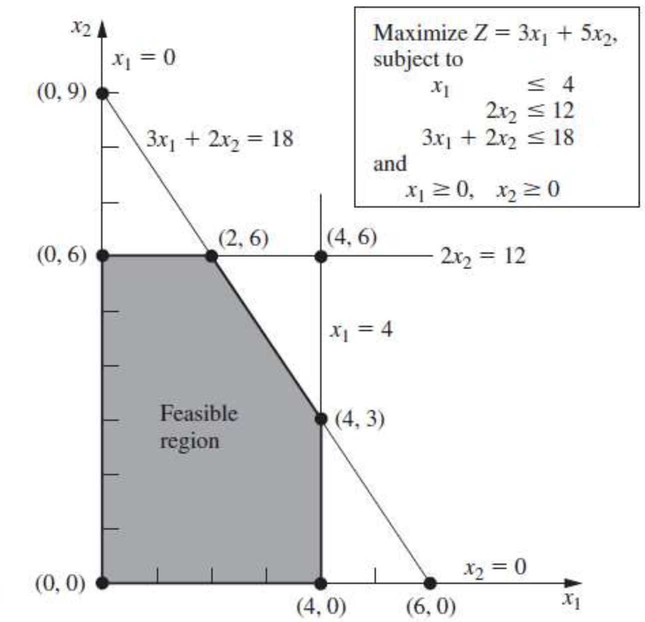
BARCELONA SCHOOL OF MANAGEMENT

In n dimensions the feasible region is a hyper polyhedron while the objective function is a plane; when it touches the polyhedron it will be in on a CPF (a corner) – or if there are more solutions, it will touch at least two CPF's (an edge or a plane)





Source (both images): Wikipedia Commons



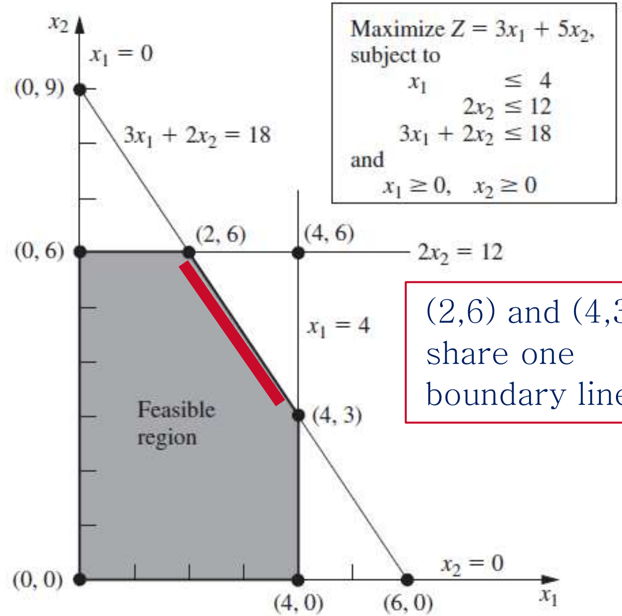
- If there is only one optimal solution this must be a CPF

→ So a brute-force strategy to find the solution is to compute Z in all CPF points

This is not what simplex does. What is the algorithm employed by simplex?

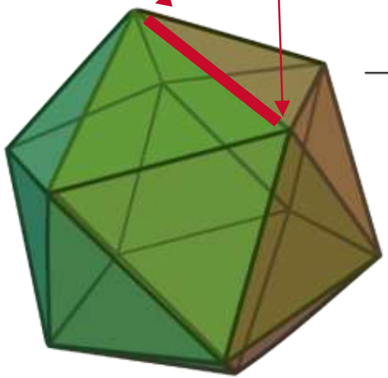
BARCELONA SCHOOL OF MANAGEMENT

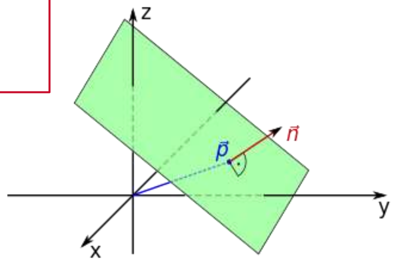
Without proof we say that two CPF are adjacent in a problem with ***n*** decision variables (*2* in the example) when the point share ***(n-1)*** constraints boundaries (*1* in this case).



(2,6) and (4,3) share one boundary line

In three dimensions these two vertices share two boundary planes





Source (both images): Wikipedia Commons

Icosahedron: twenty (eikosi=) faces/planes
Thirty edges
Twelve vertices
Faces+ vertices−2=edges (Euler's formula)

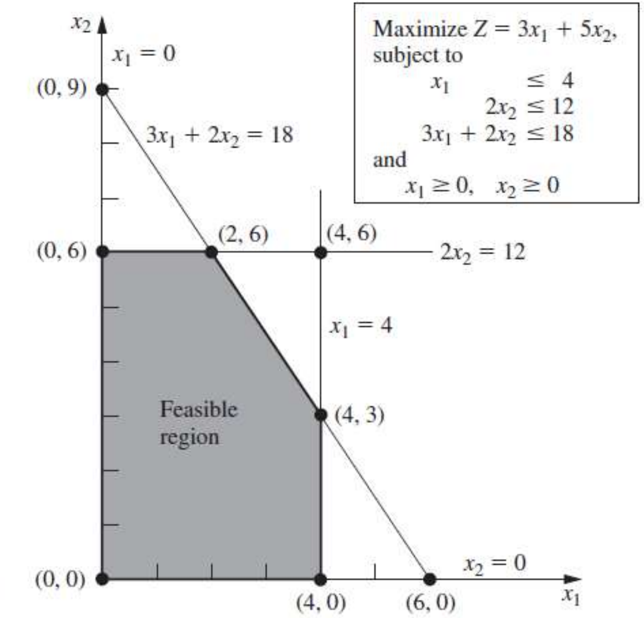


TABLE 4.1 Adjacent CPF solutions for each CPF solution of the Wyndor Glass Co. problem

CPF Solution	Its Adjacent CPF Solutions
(0, 0)	(0, 6) and (4, 0)
(0, 6)	(2, 6) and (0, 0)
(2, 6)	(4, 3) and (0, 6)
(4, 3)	(4, 0) and (2, 6)
(4, 0)	(0, 0) and (4, 3)

If the points adjacent to a given CPF all have lower Z than the given point, the given point is the optimal solution. This implies that I do not need to explore all CPF, but to follow a trajectory and systematically explore at each stage the adjacent point of my position. I stop the trajectory when all adjacent points have lower Z.

BARCELONA SCHOOL OF MANAGEMENT

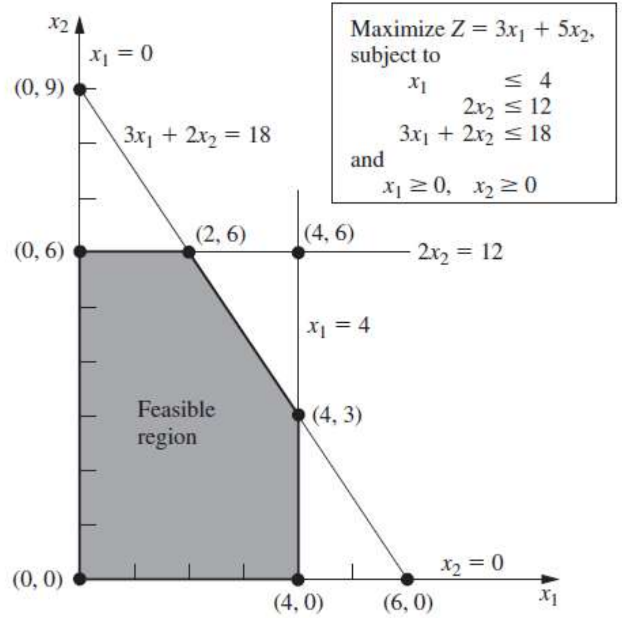
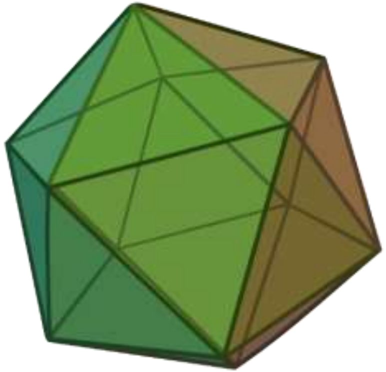


TABLE 4.1 Adjacent CPF solutions for each CPF solution of the Wyndor Glass Co. problem

CPF Solution	Its Adjacent CPF Solutions
(0, 0)	(0, 6) and (4, 0)
(0, 6)	(2, 6) and (0, 0)
(2, 6)	(4, 3) and (0, 6)
(4, 3)	(4, 0) and (2, 6)
(4, 0)	(0, 0) and (4, 3)

If the points adjacent to a given CPF all have lower Z than the given point, the given point is the optimal solution. Why?

If the points adjacent to a given CPF all have lower Z than the given point, the given point is the optimal solution. Why?



Because the solution space is convex: if you are on a peak, there cannot be a taller peak in sight

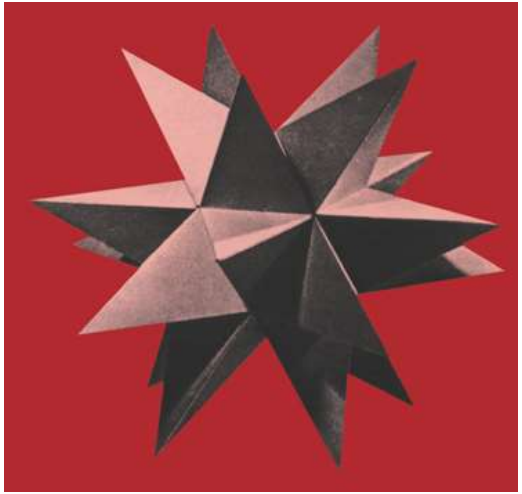
Source: https://www.istockphoto.com

BARCELONA SCHOOL OF MANAGEMENT

Because the solution space is convex: if you are on a mountain surrounded by valleys, there cannon be other mountains beyond the valleys

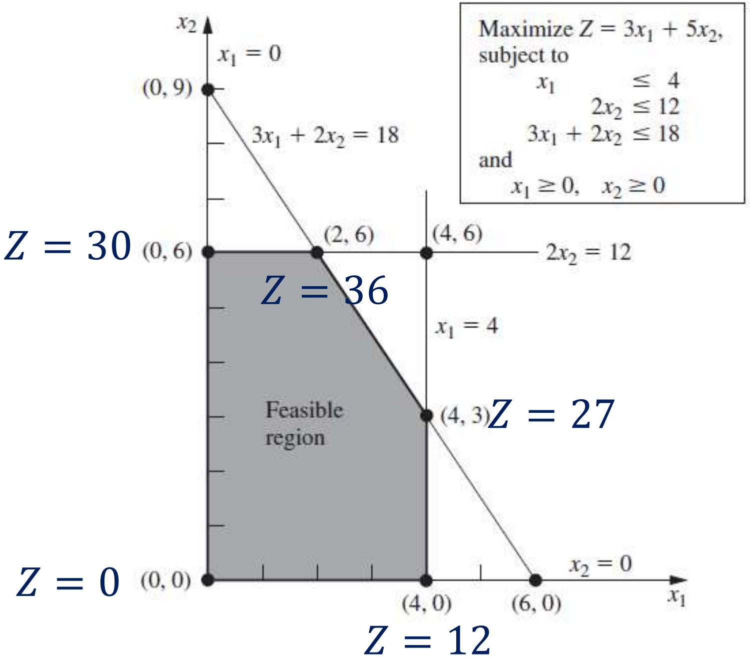
Possible solution space ←

Not a solution space →



From the cover of Proof and Refutation, of Imre Lakatos, Cambridge University Press

upf. BARCELONA SCHOOL OF MANAGEMENT



If the points adjacent to a given CPF all have lower Z than the given point, the given point is the optimal solution.

Applying this to the $n = 2$ example of the figure above, one can start from *(0,0)*, pass by *(0,6)*, and stop at (2,6) since the adjacent points of (0,6) have lower Z

Starting from (4,0) leads to the same result

BARCELONA SCHOOL OF MANAGEMENT

The nut-mix problem

The nut-mix problem of Charnes and Cooper (1953):

A manufacturer wishes to determine an optimal program for mixing three grades [A, B, D] of nuts consisting of cashews [C], hazels [H], and peanuts [P] according to the specifications and prices given in table 1. Hazels may be introduced into the mixture in any quantity, provided the specifications are met. The amounts of each nut available each day and their costs are given in table 2. Determine the pounds of each mixture that should be manufactured each day to maximize the gross return (contribution margin).

Page 94 Gass, S. I., & Assad, A. A. (2006). *An Annotated Timeline Of Operations Research: An Informal History* (1st Corrected ed. 2005. Corr. 2nd printing 2006 edition). Springer-Verlag New York Inc.

The nut-mix problem

Table 1

Mixture	Specifications	Selling price: ¢/pound
A	Not less than 50% cashews Not more than 25% peanuts	50
B	Not less than 25% cashews Not more than 50% peanuts	35
D	No specifications	25

Source: https://www.woodlandtrust.org.uk/

Hazels
(avellana)

Peanuts
(maní)

https://www.nutsforlife.com.au

Cashew
(anacardi)

Source: https://www.cashews.org

Table 2

Inputs	Capacity: pounds/day	Price: ¢/pound
C	100	65
H	60	35
P	100	25
Total	260	

BARCELONA SCHOOL OF MANAGEMENT

Table 1

Mixture	Specifications	Selling price: ¢/pound
A	Not less than 50% cashews Not more than 25% peanuts	50
B	Not less than 25% cashews Not more than 50% peanuts	35
D	No specifications	25

Hint 1

Reckon in terms of pounds per day of the three nuts type

Table 2

Inputs	Capacity: pounds/day	Price: ¢/pound
C	100	65
H	60	35
P	100	25
Total	260	

upf. BARCELONA SCHOOL OF MANAGEMENT

Table 1

Mixture	Specifications	Selling price: ¢/pound
A	Not less than 50% cashews Not more than 25% peanuts	50
B	Not less than 25% cashews Not more than 50% peanuts	35
D	No specifications	25

Hint 2

C pounds cashew/day
H pounds hazels/day
P pounds peanuts/day

C_A pounds cashew/day in A
C_B pounds cashew/day in B
…
C_P pounds peanuts/day in C
(nine variables)

Table 2

Inputs	Capacity: pounds/day	Price: ¢/pound
C	100	65
H	60	35
P	100	25
Total	260	

Check some reading material in eCampus

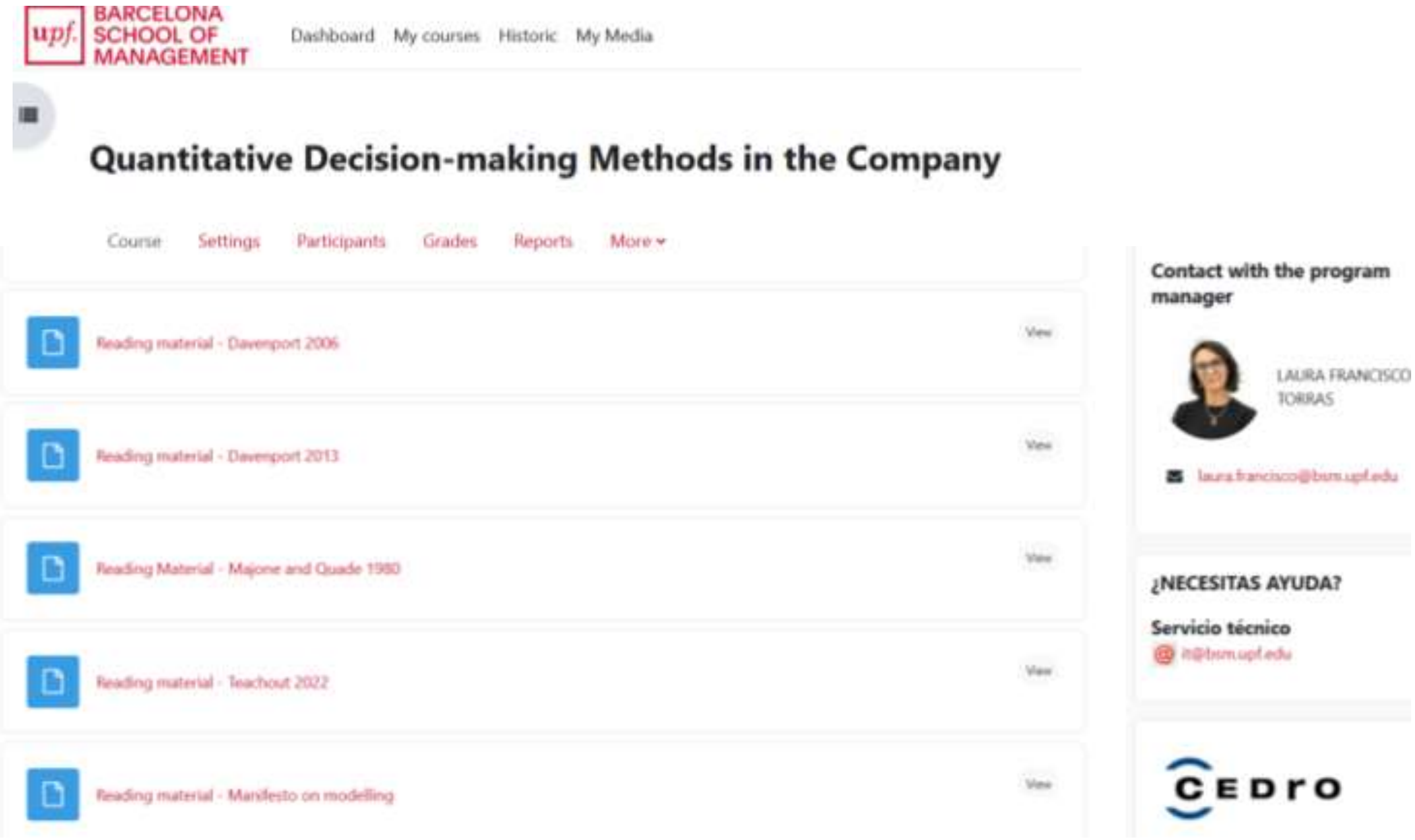
See →

Read →

See homework →

Read →

See →



Homework to be handed over at the next lesson – **handwritten**

1) Split the events in slide N. 8 starting from ‘in favour’ and ‘against’
2) Choose one Pitfall in Formulation **or** one Pitfall in Modelling from the list offered in this lecture, go to chapter 3 (from page 23) of the volume of Majone and Quade (on https://ecampus.bsm.upf.edu/) and read the relevant subsection. Write one handwritten page about what you read.
3) Consider the following model: Maximize

$$Z = 40x_1 + 50x_2$$

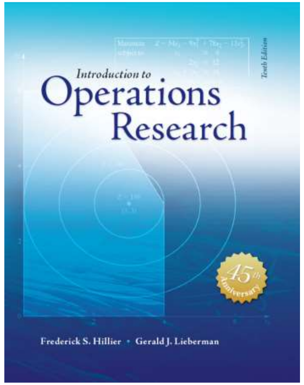
subject to

$$2x_1 + 3x_2 \geq 30$$
$$x_1 + x_2 \geq 12$$
$$2x_1 + x_2 \geq 20$$

and

$$x_1 \geq 0$$
$$x_2 \geq 0$$

Use the graphical method (paper pencil and ruler) to solve this model.
4) Write down the equations for the Nut-mix example of the previous slides without solving it.
5) Read Mann Introductory Statistics Chapter 4 Probability pages 136-176 and do exercises 4.48, 4.70, 4.76, 4.99. Please don’t give just the answer but describe the reasoning behind it.



Thank you

www.andreasaltelli.eu
https://orcid.org/0000-0003-4222-6975
@AndreaSaltelli@mstdn.social
https://www.youtube.com/channel/UCz26ZK04xchekUy4GevA3DA

upf. BARCELONA SCHOOL OF MANAGEMENT