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# A Role for Sensitivity Analysis in Presenting the Results from MCDA Studies to Decision Makers

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ABSTRACT

The aim of sensitivity analysis (SA) is to ascertain how much the uncertainty in the output of a model is influenced by the uncertainty in its input factors. An SA can be performed using different methods, which are classified according to various criteria. One possible classification is that in which global and local approaches are identified. This paper strengthens the role of global SA methods and suggests their use in the context of MCDA. Useful applications of global SA already exist in a variety of fields where numerical models are considered, e.g. in economics, engineering and chemistry. Global SA is quite different in its formulation and application from local SA, which is seen more frequently in the literature. The global sensitivity indices adopted are derived from a variance decomposition scheme: they can be estimated by alternative computational strategies, such as the extended FAST or the method proposed by Sobol'. A truly global SA is capable of apportioning the output uncertainty according to any subgroup of input factors. Hence, the output uncertainty of an MCDA can, for instance, be decomposed into a part due to uncertain model inputs and a part due to poorly defined (or variable) weights attached to the criteria. This information could be useful to the decision maker (DM) since it explains synthetically how much the assessment of an MCDA study is biased by the assessor judgements. Alternative regrouping of the uncertain input elements might shed light on other features of the problem addressed by the DM. 

KEY WORDS: sensitivity analysis; multi-criteria decision making

## 1. INTRODUCTION

It can be argued that an important element of judgement in decision making is a quantitative appreciation of the uncertainties involved, together with an indication of the likely sources of the uncertainty. While uncertainty analysis is often seen in MCDA studies, a global sensitivity analysis (SA) is still largely absent or rudimentary, especially in commercial packages for decision analysis. Indeed, the SA we have seen implemented in commercial packages is mostly in the context of linear programming (e.g. simplex method), where the purpose is to find the range of variation for the factors that is compatible with the dominant alternative (i.e. how much we can change the data of a problem, with respect to a given baseline, without misplacing the selected best alternative for that baseline). In the Bayesian framework of MCDA, SA is customarily taken to be with respect to the prior, i.e. SA is used to test the robustness of the dominating alternative with respect to uncertainties in the prior. Software packages for MCDA, such as Expert Choice or VISA, perform rather elementary SA on simple user defined models.

An ideal SA tool in support of an MCDA should be capable of taking several levels of uncertainty into account globally (i.e. over the entire problem space).

In the example discussed here we show how uncertainty can be partitioned into model input factors and weights (Saltelli and Scott, 1997). Alternative regrouping of the uncertainty in the problem could be among the so called aleatory and epistemic sources of uncertainty (Helton and Burmaster, 1996). In this latter case, aleatory would refer to all uncertainty arising from intrinsic stochastic properties of the problem (i.e. the time of occurrence of an earthquake), while the epistemic would refer to our poor knowledge of

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the physical model or of its parameters (e.g. the frequency of a Poisson law for the earthquake, see Apostolakis, 1990). The example shown in this paper represents a possible application of such a decomposition.

An SA method can be termed as global if it allows all the input factors to vary over their range of uncertainty. In contrast, a local method restricts the analysis to the neighbourhood of a working point where the factors are fixed at their nominal values and the volume of the space explored is nil. A local analysis is essentially based on differential calculus, while a global analysis involves multi-dimensional integrations over the space of variation of the input factors.

Some *desiderata* properties, such as *quantitativity* or *model independency*, are likely to be fulfilled when adopting a global approach. Let us consider these two items separately. First, a method is said to be *quantitative* when it is possible to apportion the entire output variability to the input factors without leaving some of the variability unaccounted for. A global approach is well-suited to be quantitative because the concept of decomposition of the output variance can be introduced. Secondly, a global method can be suitably designed to work on every kind of model, from linear to nonadditive and nonmonotonic. For a local approach, the concept of *quantitativity* is intrinsically meaningless, a local method being based on differential analysis. Furthermore, a local approach can only deal with perfectly linear models. New global SA methods, derived from Sobol' sensitivity indices (see Homma and Saltelli, 1996) and from the Fourier amplitude sensitivity test (FAST) (Saltelli *et al.*, 1999), appear adequate to the task for their capacity to decompose the variance of the model response quantitatively, accounting for 100% of the variance itself.

The SA measures illustrated here are due to the work of the FAST school (Cukier *et al.*, 1973, 1978; Schaibly and Schuler, 1973; Koda *et al.*, 1979) and to the original work in the field of Monte Carlo methods of Sobol' (1990). For an exhaustive review on SA techniques, see Helton (1993) and Saltelli *et al.* (1993).

Central to both the method of Sobol' and the FAST, is a variance decomposition scheme, which is also used in experimental design (Archer *et al.*, 1997). This scheme defines the methods to be *model independent*, as no assumptions about the structure of the underlying models are made (other techniques, like linear regression analysis, work only for linear models).

In variance-based methods the influence of the input factors is appreciated by way of fractional contribution to the total variance. For the simple case of a model with just three variables, the variance $D$ of the output can be written as

$$D = D_1 + D_2 + D_3 + D_{12} + D_{13} + D_{23} + D_{123}$$

where $D_i$ is a main effect (first-order term), $D_{ij}$ a two-way interaction between factors $i$ and $j$, and so on for the higher-order terms. Dividing the above by $D$, one obtains

$$1 = S_1 + S_2 + S_3 + S_{12} + S_{13} + S_{23} + S_{123}$$

where

$$S_{i_1,\ldots,i_s} = \frac{D_{i_1,\ldots,i_s}}{D}$$

For an additive model (a linear one, for instance), the sum of the first-order terms is already 1. In this case, the indices coincide with what can be obtained by regression methods (see McKay, 1995). For nonlinear models, $\Sigma_i S_i$ can be very low. While this remark stresses the importance of determining the $S_{i_1,\ldots,i_s}$ terms, it also poses a problem, as in a model with $K$ explanatory factors, the total number of terms in the decomposition is $2^K - 1$. Given that, with the method of Sobol', a separate computation (composed of a set of model evaluations) is needed for each $S_{i_1,\ldots,i_s}$, this may render the analysis impracticable. A solution is offered by the total effect terms (Homma and Saltelli, 1996), which for the example above are:

$$S_{T,1} = S_1 + S_{12} + S_{13} + S_{123}$$

$$S_{T,2} = S_2 + S_{12} + S_{23} + S_{123}$$

$$S_{T,3} = S_3 + S_{13} + S_{23} + S_{123}$$

Each of the $S_{Ti}$ can be estimated using the method of Sobol' with a single computation; further, by normalizing each $S_{Ti}$ by the sum of the $S_{Ti}$, we obtain a suitable condensed output statistics $S^*_{Ti}$,

$$S^*_{Ti} = \frac{S_{Ti}}{\sum_i S_{Ti}}$$

which can be drawn in cumulative plots as those given in Figures 2 and 3.

The $S_{Ti}$ can be evaluated either by the original method of Sobol' (1990), based on quasi-random

numbers and Monte Carlo estimation of multi-dimensional integrals, or by way of our own extension of FAST (Saltelli *et al.*, 1999). The extended FAST was seen to be computationally more efficient than the method of Sobol' (Tarantola, 1998), since the pair of indices $(S_i, S_{Ti})$ related to the factor $i$ can be estimated with a single computation. Hence, using the extended FAST, $K$ sets of model evaluations are needed to evaluate the pairs of indices (first-order and total effect) for all the factors.

The basic idea behind FAST is to identify a search-curve exploring the whole $K$-dimensional space $\Omega$ of uncertain factors. This curve should cover $\Omega$ completely, thus allowing the evaluation of one-dimensional integrals (over the curve itself) instead of multi-dimensional ones over $\Omega$. The curve is defined by the set of parametric equations

$$x_i(s) = \frac{1}{2} + \frac{1}{\pi} \arcsin(\sin(\omega_i s + \varphi_i)), \quad i = 1, \ldots, K$$

Here $s$ is a scalar variable varied over the range $-\infty < s < +\infty$, $\{\omega_i\}$ is a set of different (angular) frequencies associated with each factor $x_i$, and $\varphi_i$ is a random phase shift arbitrarily chosen in $[0, 2\pi)$. As $s$ varies, all factors change simultaneously, systematically exploring their ranges of uncertainty. The model $f(x_1, x_2, \ldots, x_K)$ may be thought of as being a one-variable function $f(s)$. Expanding $f(s)$ in a Fourier series and evaluating its frequency spectrum it is possible to estimate the total output variance $D$, by adding up the spectral components at all the frequencies. Each partial variance $D_i$ can also be estimated by an oculated choice of the set of frequencies $\{\omega_i\}$, by adding up the spectral components of $f(s)$ at the $\omega_i$ and related higher harmonics. The ratio $\hat{D}_i/\hat{D}$ is the main effect measure for factor $x_i$, i.e. $\hat{S}_i$. The extended FAST method also estimates the total effects by assigning the frequencies in a proper way. The $\hat{S}_{Ti}$ for factor $i$ can be derived from the spectral content of $f(s)$ at the lower bound of the frequency domain, while the $S_i$ is estimated as in the classical FAST by selecting $\omega_i$ in the upper region of the frequency domain. All the computational details are given in the references.

Both the method proposed by Sobol' and the extended FAST are able to perform an SA not only regarding single factors, but also groups of factors, however they are clustered. The outcome of an SA performed by groups is a total normalized effect index $S^*_{Ti}$ for each group of factors. This possibility renders the Sobol' and the extended FAST methods more attractive because the analysis may be performed at a higher descriptive level. For example, in complex models or plants with several constituting modules or subsystems, the DM could be interested in decomposing the model uncertainty according to subgroups of factors, each subgroup pertaining to the various modules or subsystems. The factors may also be grouped according to different geographical locations, or different logical types (actual factors, gridding parameters, switches, . . . ). One interesting display is likely to be one in which uncertainties arising from the weights attached to the criteria are separated from those due to the uncertain model inputs.

Besides yielding further information on the model behaviour, the results from an SA conducted by groups of factors (see Figure 3) are easier to display and interpret than those obtained by considering the factors singularly (see Figure 2), because the number of groups cannot be higher than the number of factors.

Let us illustrate the basic formalism for both the method of Sobol' and the extended FAST, in the case in which the set $\Omega$ of $K$ factors of a given model is partitioned into two groups, $u$ and $v$, such that $u + v = K$. Let $D_u$, $D_v$ and $D_{u,v}$ denote the partial variances due, respectively, to the groups $u$, $v$, and their interaction. Then

$$D = D_u + D_v + D_{u,v}$$

and, as above,

$$1 = S_u + S_v + S_{u,v}$$

$$S_{Tu} = S_u + S_{u,v}$$

$$S_{Tv} = S_v + S_{u,v}$$

and

$$1 = S^*_{Tu} + S^*_{Tv}$$

where e.g. $S^*_{Tu} = S_{Tu}/(S_{Tu} + S_{Tv})$. It is worth noting that by using the extended FAST, the two pairs of indices $(S_u, S^*_{Tu})$ and $(S_v, S^*_{Tv})$ can be estimated with two sets of model evaluations.

## 2. PROBLEM SETTING

We shall assume in the following that the object of our analysis is the 'condensed' prediction of an MCDA study, in the form of a scalar quantity $Y$ whose numerical value depends on

- a set of input factors,
- one or more models processing the information for the various pre-established criteria, and
- a series of weights attached to the criteria.

Usually there are several values of the output $Y$ corresponding to the various options being considered, but we focus on the value for a single option, formally defined by Equations (1) and (2) in the next section. We argue that if it is possible to characterize rigorously the sensitivity of $Y$, then all ancillary types of analysis (e.g. robustness with respect to the ordering of the options, and of the selection of the criteria) can be performed easily. We shall further assume that a proper quantification of the uncertainty in the model output, $Y$, is its variance, estimated within the modelling process. Thus, the problem reduces to an effective variance decomposition scheme. The outcome of an SA, i.e. the set of the $S^*_{Ti}$, could be in the form of a pie-chart, the area of the $i$th sector being proportional to the value of $S^*_{Ti}$. When performing an SA by groups, a pie-chart displaying the $S^*_{Ti}$ for each group can also be obtained. If the model output is time-dependent, such as in the example illustrated in the present work, then a pie-chart for each time point is needed. Condensed pictures for time-dependent outputs can be obtained by graphical representations like those given in Figures 2 and 3, where each pie-chart is represented by a vertical section line bounded between 0 and 1, and the areas of the various sectors within a given pie-chart can be identified by the length of the segments constituting the vertical line. Two important technicalities must be highlighted here:

1. Such variance decomposition was impossible, except for linear or at least additive models, before the new SA techniques were introduced, and
2. For complex MCDA with many uncertain inputs, the output may be hard to interpret.

Point (1) is evident to analysts familiar with regression techniques. Using for instance, as a measure of model sensitivity, the standardized regression coefficients (SRC), one can partition prediction variance quantitatively, but only for that part of it which is explained by the regression model (and is quantified by the model coefficient of determination). For nonlinear, nonadditive models this decomposable part may be uncomfortably small. Point (2) refers to the fact that a decision maker may have limited interest in a crowded diagram showing how the uncertainty in $Y$ is parcelled among several factors constituting the model (e.g. 14 as depicted in Figure 2). This problem can be easily overcome by performing an SA by groups, so that the results are easier to interpret (see Figure 3).

## 3. THE ANALYSIS

A very simple didactic test case has been selected: the Bateman equations, which describe a simple chemical or radioactive chain where each element's growth rate is directly proportional to the father concentration, and each element's decay rate $\lambda_i$ is proportional to the concentration of the element itself. The last element of the chain is assumed stable, i.e. $\lambda_N = 0$. The set of Bateman equations is:

$$\begin{aligned}
\frac{dC_1}{dt} &= -\lambda_1 C_1 \\
\frac{dC_2}{dt} &= -\lambda_2 C_2 + \lambda_1 C_1 \\
&\vdots \\
\frac{dC_N}{dt} &= \lambda_{N-1} C_{N-1}
\end{aligned} \tag{1}$$

whose solution is

$$C_i(t) = \sum_{m=1}^{i} \left\{ C_m^0 \prod_{r=m, m\neq i}^{i-1} \left[ \lambda_r \sum_{n=m}^{i} \left( \frac{e^{-\lambda_n t}}{\prod_{l=m, m\neq i, l\neq n}^{i} (\lambda_l - \lambda_n)} \right) \right] \right\}$$

Here $\lambda_i$ and $C_i^0$ are, respectively, the decay rate and the initial concentration of species $i$. The model output has been defined as

$$Y(t) = \sum_{i=1}^{N} w_i C_i(t) \tag{2}$$

For each time point, in a prescribed range, we have computed the concentrations, $C_i(t)$, for a system of $N$ species. The concentrations have then been averaged by means of weights $w_i$, as if each species $i$ were a different risk criterion, and the weights were to reflect the assessor's belief on the

Table I. List of input factors and their pdfs for the test case defined by Equations (1) and (2)

| Factor | Definition | Range | Units |
|---|---|---|---|
| $w_1$ | Weight 1 | /1, 10/ | risk/mole |
| $w_2$ | Weight 2 | /100, 200/ | risk/mole |
| $w_3$ | Weight 3 | /1, 5/ | risk/mole |
| $w_4$ | Weight 4 | /1, 10/ | risk/mole |
| $w_5$ | Weight 5 | /1, 10/ | risk/mole |
| $C_1^0$ | Initial concentration 1 | /0, 15/ | moles |
| $C_2^0$ | Initial concentration 2 | /0, 10/ | moles |
| $C_3^0$ | Initial concentration 3 | /0, 10/ | moles |
| $C_4^0$ | Initial concentration 4 | /0, 10/ | moles |
| $C_5^0$ | Initial concentration 5 | /0, 10/ | moles |
| $\lambda_1$ | Decay rate 1 | $/1.24\times10^{-8}$, $8.24\times10^{-5}/$ | 1/s |
| $\lambda_2$ | Decay rate 2 | $/1.0\times10^{-7}$, $1.0\times10^{-4}/$ | 1/s |
| $\lambda_3$ | Decay rate 3 | $/9.0\times10^{-7}$, $1.0\times10^{-4}/$ | 1/s |
| $\lambda_4$ | Decay rate 4 | $/2.0\times10^{-6}$, $3.0\times10^{-3}/$ | 1/s |

danger/importance of the criterion. The model output, $Y$, can be considered as an index of the risk to the population due to the system of $N$ species. $Y$ is driven by the following uncertain parameters:

- $N$ weights $w_i$
- $N$ initial concentrations $C_i^0$
- $N-1$ decay rates $\lambda_i$

where $N$ ($N=5$ in our example) indicates the number of species in the chain, so that the system has 14 uncertain input factors. The weights are typically given by expert judgement.

Both the initial concentrations and the decay rates are supposed to be poorly known factors, which constitute our parametric uncertainty. Ranges and distributions for the 14 factors are depicted in Table I.

A first analysis was conducted by evaluating the first-order (no interactions) sensitivity indices for all 14 factors (rates, concentrations, weights). It can be noted that the first-order indices (Figure 1) add up to 1 for times up to about $5\times10^5$ s, identifying the time range where the model is fully additive. For this time range even an analysis based on classical FAST, or perhaps on the SRCs, would be adequate. At later times, the sum drops to values below 0.6, pointing out that a large fraction of the output variance can not be explained using only the main effect indices. The model loses its additive properties there and this is due to the intervention of the decay parameters. It should be stressed that we are using a very simple model; in real applications, the additivity of models can be much lower (Saltelli *et al.*, 1999).

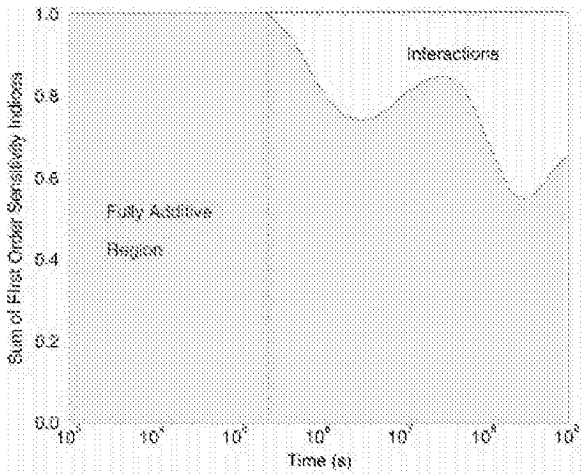


Figure 1. The extended FAST has been employed to evaluate the first-order indices for all 14 factors as a function of time. The computation requires $257\times14=3598$ model evaluations (the same set can be used to evaluate the total indices). The sum of the first-order indices is displayed. It can be noted that the model is perfectly additive up to $2\times10^5$ s, hence first-order indices yield all the information. Afterwards, the sum drops down to 0.6–0.8 due to nonadditive effects, thus implying the use of the total indices in order to perform a quantitative analysis

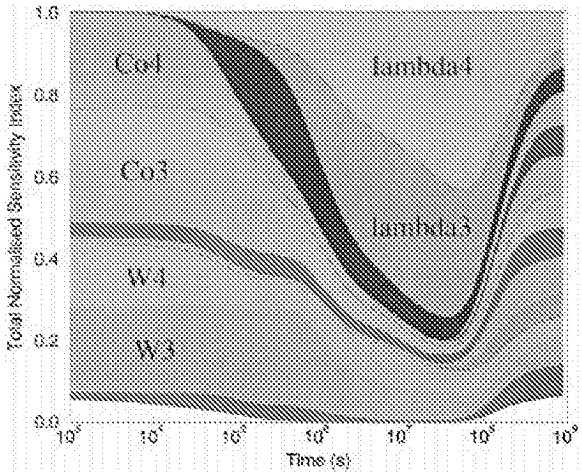


Figure 2. The extended FAST has been employed to evaluate the total normalized sensitivity indices for all 14 factors as a function of time. The computation requires $257 \times 14 = 3598$ model evaluations. The total normalized indices are displayed in a cumulative plot. It can be noted that the most predominating factors are W3, W4, $C_3^0$ and $C_4^0$ up to $10^6$ s; $\lambda_3$ and $\lambda_4$ can be considered as the driving factors in the time interval $(10^7-10^8)$s, (they account for about 70% of the output variance); beyond $10^8$ s, when the transient ends and the model loses its additivity properties, all the factors tend to become equally important

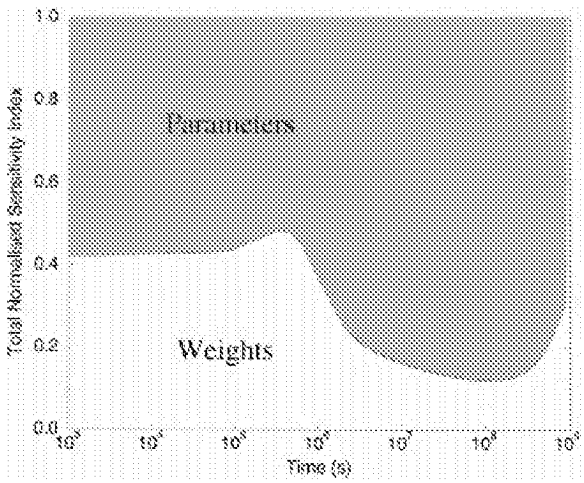


Figure 3. The extended FAST has been employed to evaluate the total normalized sensitivity indices for two groups of factors (model parameters and weights) as a function of time. The computation requires $257 \times 2 = 514$ model evaluations. The total indices for the two groups are displayed in a cumulative plot. It can be noted that after $10^6$ s the overall importance due to the weights oscillates between 20 and 30%

An analysis of the total normalized sensitivity indices has then been performed (Figure 2); from this figure it is clear why the model is additive till $5 \times 10^5$ s, as this is the area where weights and initial concentrations dominate the total variance. Even for this very simple model with just 14 uncertain factors, Figure 2 is difficult to read (only the most important factors have been labelled). By regrouping the terms as described in the previous section, we obtain Figure 3. This figure has two important properties:

1. It is quantitative, i.e. the fractional contribution to the total variance is not the result of an approximation; 100% of the output variance is accounted for.
2. The figure has a clear appeal for simplicity. Two logically distinct classes of uncertainty have been effectively evaluated and displayed.

## 4. CONCLUSIONS

In summary, the extended FAST enables the analyst to display concisely the information needed by a decision maker. The measure is quantitative and exhaustive; it includes all contributions to the output variance, i.e. both additive and nonadditive effects. The measure shows clearly and synthetically how much the assessment is biased by uncertainties in this or that domain. In the example offered here, we investigated the relative importance of the assessor belief, i.e. we showed (Figure 3) that the analysis is more robust for times after $10^6$ s, where the overall contribution of the weights falls below 20–30%. It might be argued, in conclusion, that the variance decomposition as shown in the present note could become a prescription for MCDA studies and for the presentation of their results.

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